

A local search 2.917-approximation algorithm for duo-preservation string mapping*

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Abstract

We study the *maximum duo-preservation string mapping* (MAX-DUO) problem, which is the complement of the well studied *minimum common string partition* (MCSP) problem. Both problems have applications in many fields including text compression and bioinformatics. Motivated by an earlier local search algorithm, we present an improved approximation and show that its performance ratio is no greater than $35/12 < 2.917$. This beats the current best 3.25-approximation for MAX-DUO. The performance analysis of our algorithm is done through a complex yet interesting amortization. Two lower bounds on the locality gap of our algorithm are also provided.

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1 Introduction

The *minimum common string partition* (MCSP) problem is a well-studied problem in computer science, with applications in the fields such as text compression and bioinformatics. MCSP was first introduced by Goldstein *et al.* [13] as follows: Consider two length- n strings $A = (a_1, a_2, \dots, a_n)$ and $B = (b_1, b_2, \dots, b_n)$ over some alphabet Σ , such that B is a permutation of A . A *partition* of A , denoted as \mathcal{P}_A , is a multi-set of substrings whose concatenation in a certain order becomes A . The number of substrings in \mathcal{P}_A is the *cardinality* of \mathcal{P}_A . The MCSP problem asks for a minimum cardinality partition \mathcal{P}_A of A that is also a partition of B . When every letter of the alphabet Σ occurs at most k times in each of the two strings, the restricted version of MCSP is denoted as k -MCSP.

The MCSP problem is NP-hard and APX-hard even when $k = 2$ [13]. Several approximation algorithms [8, 9, 10, 13, 15, 16] have been presented since 2004. The current best result is an $O(\log n \log^* n)$ -approximation for the general MCSP and an $O(k)$ -approximation for k -MCSP. On the other hand, MCSP is proved to be *fixed parameter tractable* (FPT), with respect to k and/or to the cardinality of the optimal partition [11, 14, 5, 6].

Given a string, an ordered pair of consecutive letters is called a *duo* [13]; a length- ℓ substring in a partition *preserves* $\ell - 1$ duos of the given string. The complementary objective

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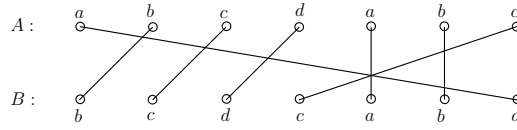
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to that of MCSP is to maximize the number of duos preserved in the common partition, which is referred to as the *maximum duo-preservation string mapping* (MAX-DUO) problem by Chen *et al.* [7] and is our target problem in this paper. Analogously, k -MAX-DUO is the restricted version of MAX-DUO when every letter of the alphabet Σ occurs at most k times in each of the two given strings.

We next give a graphical view on a common partition of the two given strings $A = (a_1, a_2, \dots, a_n)$ and $B = (b_1, b_2, \dots, b_n)$. Construct a bipartite graph $G = (A, B, E)$, where the vertices of A (B , respectively) are a_1, a_2, \dots, a_n in order (b_1, b_2, \dots, b_n in order, respectively) and there is an edge between a_i and b_j if they are the same letter. A common partition \mathcal{P} of the strings A and B one-to-one corresponds to a perfect matching M in the graph G (see Fig. 1.1 for an example), and the number of duos preserved by the partition is exactly the number of pairs of *parallel* edges in the matching; if both $(a_i, b_j), (a_{i+1}, b_{j+1}) \in E$, then they form a pair of parallel edges.



■ **Figure 1.1** An instance of the MAX-DUO problem with two strings $A = (a, b, c, d, a, b, c)$ and $B = (b, c, d, c, a, b, a)$, and a common partition $\{a, bcd, ab, c\}$ that preserves three duos (b, c) , (c, d) and (a, b) , corresponding to the perfect matching shown in the figure.

Along with MAX-DUO, Chen *et al.* [7] introduced the *constrained maximum induced subgraph* (CMIS) problem, in which one is given an m -partite graph $G = (V_1, V_2, \dots, V_m, E)$, with each V_i having n_i^2 vertices arranged in an $n_i \times n_i$ matrix, and the goal is to select n_i vertices of each V_i in different rows and different columns such that the induced subgraph contains the maximum number of edges. The restricted version of CMIS when $n_i \leq k$ for all i is denoted as k -CMIS.

For an instance of the MAX-DUO problem, one can first set m to be the number of distinct letters in the string A , set n_i to be the number of occurrences of the i -th distinct letter, and the (s, t) -vertex in the $n_i \times n_i$ matrix “means” mapping the s -th occurrence of the i -th distinct letter in the string A to its t -th occurrence in the string B ; and then set an edge connecting a vertex of V_i and a vertex of V_j if the two vertices together preserve a duo. This way, the MAX-DUO problem becomes a special case of the CMIS problem, and furthermore the k -MAX-DUO is a special case of the k -CMIS.

Chen *et al.* [7] presented a k^2 -approximation for k -CMIS and a 2-approximation for 2-CMIS, based on linear programming and a randomized rounding. These imply that k -MAX-DUO can also be approximated within a ratio of k^2 and 2-MAX-DUO can be approximated within a ratio of 2.

Continuing on the graphical view as shown in Fig. 1.1 on a common partition of the two given strings A and B , we can construct another graph $H = (V, F)$ in which every vertex of V corresponds to a pair of parallel edges in the bipartite graph $G = (A, B, E)$, and two vertices of V are adjacent if the two pairs of parallel edges of E cannot co-exist in any perfect matching of G (called *conflicting*, which can be determined in constant time, see Section 2). This way, a set of duos that can be preserved by some perfect matching of G (called *compatible*, see Section 2) one-to-one corresponds to an independent set of H [13, 3]. Therefore, the MAX-DUO problem can be cast as a special case of the well-known *maximum independent set* (MIS) problem [12]; in particular, Boria *et al.* [3] showed that

an instance of k -MAX-DUO translates to a graph with the maximum degree $\Delta \leq 6(k-1)$. Since MIS can be approximated arbitrarily close to $(\Delta+3)/5$ [1], k -MAX-DUO can now be better approximated within a ratio of $(6k-3)/5 + \epsilon$, for any $\epsilon > 0$, using the same algorithm. Especially, 2-MAX-DUO and 3-MAX-DUO can be approximated within a ratio of $1.8 + \epsilon$ and $3 + \epsilon$, respectively. Boria *et al.* [3] also proved that MAX-DUO is APX-hard, even when $k = 2$.

In Section 2, we will construct another bipartite graph for an instance of the MAX-DUO problem, and thus cast MAX-DUO as a special case of the *maximum compatible bipartite matching* (MCBM) problem. Such a reduction was first shown by Boria *et al.* [3], who presented a 4-approximation for the MCBM problem, implying that MAX-DUO can also be approximated within a ratio of 4. Boria *et al.* [2] also used this reduction, with the word *consecutive* in place of *compatible*, to present a local search 3.5-approximation for the MCBM problem.

Most recently, Brubach [4] presented a 3.25-approximation for the MAX-DUO based on a novel *combinatorial triplet matching*. This 3.25-approximation is the current best for the general MAX-DUO problem.

The basic idea in the local search 3.5-approximation for the MCBM problem by Boria *et al.* [2] is to swap one edge of the current matching out for two compatible edges, thus to increase the size of the matching till a local optimum is reached. The performance ratio 3.5 is shown to be tight. We extend this idea to allow swapping five edges of the current matching out for six compatible edges, and we also allow a new operation of swapping five edges of the current matching out for five compatible edges if the number of *singleton edges* (to be defined in Section 2) is strictly decreased. Through a complex yet interesting amortized analysis, we prove that our local search heuristics has an approximation ratio of at most $35/12 < 2.917$, which improves the current best 3.25-approximation algorithm and breaks the barrier of 3. In a companion paper [17], we propose a $(1.4 + \epsilon)$ -approximation for the 2-MAX-DUO; thus together we improve all the current best approximability results.

The rest of the paper is organized as follows: We provide some preliminaries in Section 2, including the formal description of the MCBM problem and the terminologies and notations to be used throughout the paper. Our local search heuristics is presented in Section 3. In Section 4, we analyze the approximation ratio of our heuristics through amortization. In Section 5, we show a lower bound of $13/6 > 2.166$ on the locality gap of our algorithm for the MCBM problem, and a lower bound of $5/3 > 1.666$ on the locality gap of our algorithm for the MAX-DUO problem. We conclude the paper in Section 6.

2 Preliminaries

Recall that in an instance of the MAX-DUO problem, we have two length- n strings $A = (a_1, a_2, \dots, a_n)$ and $B = (b_1, b_2, \dots, b_n)$ such that B is a permutation of A . We use $d_i^A = (a_i, a_{i+1})$ and $d_i^B = (b_i, b_{i+1})$ to denote the i -th duo of A and B , respectively, for $i = 1, 2, \dots, n-1$; and $D^A = \{d_1^A, d_2^A, \dots, d_{n-1}^A\}$ and $D^B = \{d_1^B, d_2^B, \dots, d_{n-1}^B\}$. We construct a bipartite graph $G = (D^A, D^B, E)$, where there is an edge $e_{i,j}$ connecting d_i^A and d_j^B if $a_i = b_j$ and $a_{i+1} = b_{j+1}$, suggesting that the duo d_i^A is preserved if the edge $e_{i,j} = (d_i^A, d_j^B)$ is selected into the solution matching. (See Fig. 2.1a for the bipartite graph constructed from the two strings shown in Fig. 1.1.) Note that selecting the edge $e_{i,j}$ rules out all the other edges incident at d_i^A and all the other edges incident at d_j^B , and some more edges described in the next paragraph.

Formally, the two edges $e_{i,j}$ and $e_{i',j'}$ with $j \neq j'$ are called *adjacent*, and they are

conflicting since they cannot be both selected into a feasible solution matching. Similarly, two adjacent edges $e_{i,j}$ and $e_{i',j}$ with $i \neq i'$ are conflicting. The two edges $e_{i,j}$ and $e_{i+1,j+1}$ are called *parallel*; while the two edges $e_{i,j}$ and $e_{i+1,j'}$ with $j' \neq j, j+1$ are called *neighboring*. Two neighboring edges are conflicting too since they cannot be both selected. Similarly, the two edges $e_{i,j}$ and $e_{i',j+1}$ with $i' \neq i, i+1$ are neighboring and conflicting. Any two unconflicting edges are said *compatible* to each other, and a *compatible* set of edges contains edges that are pairwise compatible, which is consequently a feasible solution matching (called a *compatible matching*). (See Fig. 2.1b for a compatible matching found in the bipartite graph in Fig. 2.1a.) The goal of the *maximum compatible bipartite matching* (MCBM) problem is to find a maximum cardinality compatible matching in the bipartite graph $G = (D^A, D^B, E)$.

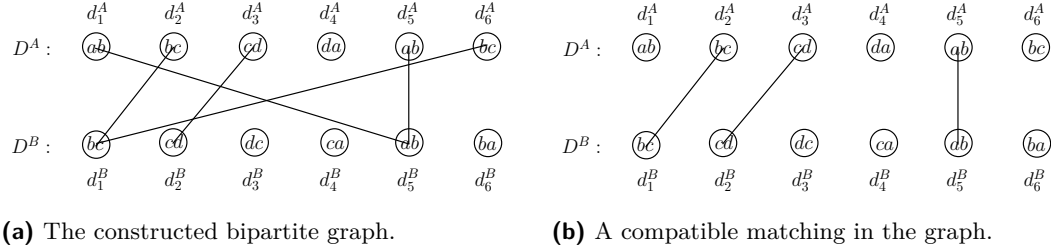


Figure 2.1 A bipartite graph $G = (D^A, D^B, E)$ constructed from the two strings $A = (a, b, c, d, a, b, c)$ and $B = (b, c, d, c, a, b, a)$, and a compatible matching in G containing three edges $e_{2,1}, e_{3,2}, e_{5,5}$.

Clearly, the bipartite graph $G = (D^A, D^B, E)$ in the MCBM problem does not have to be constructed out of two given strings in the MAX-DUO problem, and therefore MAX-DUO is a special case of MCBM. Nevertheless, when restricted to MAX-DUO, the cardinality of a compatible matching is exactly the number of duos preserved by the matching. An edge in a compatible matching M is called *singleton* if it is not parallel to any other edge in the matching. This way, the matching M is partitioned into two parts: $s(M)$ containing all the singleton edges and $p(M)$ containing all the parallel edges. A series of pairs of parallel edges $e_{i,j}, e_{i+1,j+1}, \dots, e_{i+p,j+p}$, for some $p \geq 2$, is referred to as *consecutive parallel edges*.

Except towards the end we show a lower bound on the locality gap of our local search heuristics for the MAX-DUO problem, all discussion in the sequel is on the MCBM problem. The obtained approximability results on the MCBM problem also apply to the MAX-DUO problem.

► **Observation 2.1.** Any edge $e_{i,j} \in E$ can be conflicting with at most 6 edges that are pairwise compatible, which are $e_{i,j'}$, $e_{i-1,j''-1}$, $e_{i+1,j''' + 1}$, $e_{i',j}$, $e_{i''-1,j-1}$, $e_{i''' + 1,j+1}$ incident at $d_{i-1}^A, d_i^A, d_{i+1}^A, d_{j-1}^B, d_j^B, d_{j+1}^B$, respectively, where none of i', i'', i''' can be i and none of j', j'', j''' can be j .

We remark that in Observation 2.1 by “at most”, some of the six edges could be void, that is, non-existent in E ; also, when $e_{i,j'}$ and $e_{i-1,j''-1}$ both present, then they have to be parallel suggesting that $j' = j''$ (the same applies to $e_{i,j'}$ and $e_{i+1,j''' + 1}$, $e_{i',j}$ and $e_{i''-1,j-1}$, $e_{i',j}$ and $e_{i''' + 1,j+1}$).

In the sequel, in general, the subscript of a vertex of D^A has an i or h , and the subscript of a vertex of D^B has a j or ℓ .

3 A local search heuristics \mathcal{LS}

Given a bipartite graph $G = (D^A, D^B, E)$, the 3.5-approximation algorithm presented by Boria *et al.* [2] starts with an arbitrary maximal compatible matching, iteratively seeks swapping one edge in the current matching out for two compatible edges, and terminates when the expansion by such swapping is impossible.

Our local search heuristics is an extension of the above algorithm, to iteratively apply two different swapping operations to increase the size of the matching and to decrease the number of singleton edges in the matching, respectively. We present the heuristics in details in the following. Note that we also start with an arbitrary maximal compatible matching, which by Observation 2.1 can be obtained in $O(n^2)$ -time, where n is the number of vertices in one side of the bipartite graph (or more precisely, $|D^A| = |D^B| = n - 1$).

Let M denote the current compatible matching in hand. For any edge $e_{i,j} \in M$, let $C(e_{i,j})$ be the set of all the edges of E conflicting with $e_{i,j}$; then ($q = -1, 0, +1$ in the following set unions)

$$C(e_{i,j}) = \bigcup_{q=-1}^{+1} \{e_{i+q,j'+q} \in E \mid j' \neq j\} \cup \bigcup_{q=-1}^{+1} \{e_{i'+q,j+q} \in E \mid i' \neq i\}. \quad (1)$$

Clearly, $|C(e_{i,j})| \leq 6(n-1)$. Recall that $|E| \in O(n^2)$. We have the following observation, which essentially narrows down the candidate edges for swapping with the edge $e_{i,j}$.

► **Observation 3.1.** *For a maximal compatible matching M and an edge $e_{i,j} \in M$, the edges compatible with all the edges of $M - \{e_{i,j}\}$ must be in $C(e_{i,j}) \cup \{e_{i,j}\}$.*

We next describe the two different swapping operations. Both of them apply to a maximal compatible match M . One operation is to replace five edges of M by six edges, denoted as REPLACE-5-BY-6, thus to increase the size of the matching; and the other operation is to replace five edges of M by five edges with the resulted matching having strictly less singleton edges, denoted as REDUCE-5-BY-5. Note that in each iteration, the operation REDUCE-5-BY-5 applies only when the operation REPLACE-5-BY-6 fails to expand the current matching M .

3.1 Operation REPLACE-5-BY-6

The operation REPLACE-5-BY-6 seeks to expand the current maximal compatible matching M by swapping five edges of M out for six compatible edges. It does so by scanning all size-5 subsets of M and terminates at a successful expansion. If no such expansion is possible, it also terminates but without making any change to the matching M .

Let $X = \{e_1, e_2, \dots, e_5\}$ be a subset of M (in the special case where $|M| \leq 5$, we seek for a compatible matching of size $|M| + 1$ directly by an exhaustive search). The operation composes a set $E' = X \cup C(X)$, where $C(X)$ contains all the edges each conflicting with an edge of X but compatible with (all the edges of) $M - X$; it then checks every size-6 subset X' of E' for compatibility and, if affirmative, swaps X out for X' to expand M .

Recall that $|M| < n$. The number of size-5 subsets of M is $O(n^5)$. For each size-5 subset X , composing the set E' takes $O(n^2)$ time and $|E'| < 30n$. It follows that the number of size-6 subsets of E' is $O(n^6)$. Lastly, checking the compatibility of each size-6 subset X' takes $O(1)$ time. Therefore, the time complexity of the operation REPLACE-5-BY-6 is $O(n^{11})$.

3.2 Operation REDUCE-5-BY-5

From Equation 1, one sees that given a maximal compatible matching M , a pair of parallel edges of M are expected to conflict much less edges outside of M than two singleton edges of M do. This hints that for two compatible matchings of the same cardinality, the one with more parallel edges more likely can be expanded, and motivates the new operation REDUCE-5-BY-5.

When the operation REPLACE-5-BY-6 fails to expand the current maximal compatible matching M , the operation REDUCE-5-BY-5 seeks to decrease the number of singleton edges in M , by swapping five edges of M out for five compatible edges. Similarly, it does so by scanning all size-5 subsets of M , and terminates at a successful reduction. If no such reduction is possible, it also terminates but without making any change to the matching M .

Recall that M is partitioned into $p(M)$ and $s(M)$, containing all the parallel edges and all the singleton edges, respectively. Let $X = \{e_1, e_2, \dots, e_5\}$ be a subset of M (in the special case where $|M| \leq 5$, we seek for a compatible matching of the same size but containing strictly less singleton edges directly by an exhaustive search). The operation composes a set $E' = X \cup C(X)$, where $C(X)$ contains all the edges each conflicting with an edge of X but compatible with $M - X$; it then checks every size-5 subset X' of E' for compatibility and subsequently checks whether $|s(M - X \cup X')| < |s(M)|$, if both affirmative, swaps X out for X' to reduce the number of singleton edges in M .

For the time complexity of the operation REDUCE-5-BY-5, similarly we recall that $|M| < n$. Partitioning M into $p(M)$ and $s(M)$ takes at most $O(n^2)$ time. There are $O(n^5)$ size-5 subsets of M . For each such size-5 subset X , composing the set E' takes $O(n^2)$ time and $|E'| < 30n$. It follows that the number of size-5 subsets of E' is $O(n^5)$. Lastly, checking the compatibility of each size-5 subset X' takes $O(1)$ time and counting the singleton edges of $M - X \cup X'$ can be done in $O(n)$ time. Therefore, the time complexity of the operation REDUCE-5-BY-5 is $O(n^{11})$ too.

3.3 The local search heuristics \mathcal{LS}

Our local search heuristics is iterative. The compatible matching M is initialized to \emptyset .

At the beginning of each iteration, we greedily expand the current compatible matching M to the maximal, by adding one edge at a time. Next, with the current maximal compatible matching M , the operation REPLACE-5-BY-6 is applied to expand M . If successful, the iteration ends. Otherwise, M is not modified by the operation REPLACE-5-BY-6 and the operation REDUCE-5-BY-5 is applied to reduce the number of singleton edges in M . If successful, the iteration ends; otherwise the entire algorithm terminates and returns the current M as the solution.

Clearly, the step of greedy expansion takes $O(n^2)$ time. The running time of the rest of the iteration is $O(n^{11})$, which is dominant.

Note that every iteration, except the last, either increases the cardinality of the compatible matching or decreases the number of the singleton edges in the compatible matching. We thus conclude that there are $O(n^2)$ iterations in the entire algorithm, which we denote as \mathcal{LS} . It follows that the time complexity of the algorithm \mathcal{LS} is $O(n^{13})$. We state this result in the following theorem.

► **Theorem 3.1.** *The time complexity of the local search heuristics \mathcal{LS} for the MCBM problem is $O(n^{13})$, where n is the number of vertices in one side of the bipartite graph.*

4 Approximation ratio analysis for the heuristics \mathcal{LS}

We analyze the performance ratio of the heuristics \mathcal{LS} through *amortization*. The main result is to prove that the heuristics \mathcal{LS} is a 35/12-approximation for the MCBM problem, and thus it is also a 35/12-approximation for the MAX-DUO problem.

4.1 The amortization scheme

Let M^* be the optimal compatible matching to the MCBM problem and $\text{OPT} = |M^*|$, and M be the maximal compatible matching returned by the algorithm \mathcal{LS} and $\text{SOL} = |M|$. We partition M into $s(M)$ and $p(M)$. (In the sequel, notations with a superscript $*$ are associated with M^* ; notations without a superscript are associated with M . In general, the subscript of a vertex of D^A has an i or h , and the subscript of a vertex of D^B has a j or ℓ .)

In the amortization scheme, we assign one token to each edge $e^* \in M^*$, and thus the total amount of tokens is OPT . The edge e^* will be conflicting to a number of edges of M (including the case where e^* is in M , then e^* is conflicting to itself only); it then splits the token evenly and distributes a fraction to every conflicting edge of M . To the end, the total amount of tokens received by all the edges of M is exactly OPT . Our main task is to estimate an upper bound (which is expected to be 35/12) on the amount of tokens received by an edge of M , thereby to give a lower bound on SOL .

Formally, we define the function $\tau(e \leftarrow e^*) \geq 0$ to be the amount of token $e^* \in M^*$ gives to $e \in M$. For the edge $e^* \in M^*$, let $C(e^*) \subseteq M$ be the subset of edges of M conflicting with e^* , and for the edge $e \in M$, let $C^*(e) \subseteq M^*$ be the subset of edges of M^* conflicting with e . From the maximality, we know that both $|C(e^*)|, |C^*(e)| \geq 1$, for any e^*, e . Then, $\tau(e \leftarrow e^*) = \frac{1}{|C(e^*)|}$, if $e \in C(e^*)$; or otherwise $\tau(e \leftarrow e^*) = 0$. The total amount of tokens $e \in M$ receives is denoted as

$$\omega(e) := \sum_{e^* \in C^*(e)} \frac{1}{|C(e^*)|}, \forall e \in M. \quad (2)$$

And we have

$$\text{OPT} = \sum_{e \in M} \omega(e) \leq \max_{e \in M} \omega(e) \cdot \text{SOL}.$$

Therefore, the quantity $\max_{e \in M} \omega(e)$ is an upper bound on the performance ratio of the algorithm \mathcal{LS} . We thus aim to estimate $\max_{e \in M} \omega(e)$. In the following, we will see that $\max_{e \in M} \omega(e) = 10/3$, which is larger than our target ratio 35/12. We then switch to enumerate all possible cases where an edge e has $\omega(e) \geq 3$ and amortize some fraction of its token to certain provably existing edges e' with $\omega(e') < 3$. In other words, we will estimate the average value of $\omega(\cdot)$ for all the edges of M , denoted as $\overline{\omega(e)}$, and prove an upper bound (which is shown to be 35/12) on $\overline{\omega(e)}$ that is also an upper bound on the performance ratio of the algorithm \mathcal{LS} .

To this purpose, we may assume without loss of generality that $M \cap M^* = \emptyset$ since their $\omega(\cdot)$'s are all 1. According to Observation 2.1 in Section 2, we have $|C^*(e)| \leq 6$ and $|C(e^*)| \leq 6$ for any $e \in M$ and $e^* \in M^*$. Consider an arbitrary edge $e_{i,j} \in M$, we have

$$C^*(e_{i,j}) = \{e_{i-1,j''-1}^*, e_{i,j'}^*, e_{i+1,j''' + 1}^*, e_{i''-1,j-1}^*, e_{i',j}^*, e_{i''' + 1,j+1}^*\},$$

where $e_{i,j'}^*$ ($e_{i-1,j''-1}^*, e_{i+1,j''' + 1}^*$, respectively) denotes the edge of M^* incident at d_i^A (d_{i-1}^A, d_{i+1}^A , respectively), if it exists, or otherwise it is a void edge; $e_{i',j}^*$ ($e_{i''-1,j-1}^*, e_{i''' + 1,j+1}^*$,

respectively) denotes the edge of M^* incident at d_j^B (d_{j-1}^B, d_{j+1}^B , respectively), if it exists, or otherwise it is a void edge; and none of i', i'', i''' can be i and none of j', j'', j''' can be j . (It is important to point out that $C^*(e_{i,j})$ does not necessarily contain 6 edges, due to the possible void edges.) We partition $C^*(e_{i,j})$ into two parts $C^*(e_{i,\cdot})$ and $C^*(e_{\cdot,j})$:

$$\begin{aligned} C^*(e_{i,\cdot}) &= \{e_{i-1,j''-1}^*, e_{i,j'}^*, e_{i+1,j''' + 1}^*\}, \\ C^*(e_{\cdot,j}) &= \{e_{i''-1,j-1}^*, e_{i',j}^*, e_{i''' + 1,j+1}^*\}. \end{aligned}$$

(Again, each of $C^*(e_{i,\cdot})$ and $C^*(e_{\cdot,j})$ does not necessarily contain 3 edges, due to the possible void edges.) We extend the function notation to let $\tau(e_{i,j} \leftarrow C^*(e_{i,j}))$ be the multi-set of the $\tau(e_{i,j} \leftarrow e^*)$ values, where $e^* \in C^*(e_{i,j})$, that is,

$$\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) = \left\{ \frac{1}{|C(e^*)|} \mid e^* \in C^*(e_{i,\cdot}) \right\}, \quad (3)$$

$$\tau(e_{i,j} \leftarrow C^*(e_{\cdot,j})) = \left\{ \frac{1}{|C(e^*)|} \mid e^* \in C^*(e_{\cdot,j}) \right\}, \quad (4)$$

$$\tau(e_{i,j} \leftarrow C^*(e_{i,j})) = \tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) \cup \tau(e_{i,j} \leftarrow C^*(e_{\cdot,j})). \quad (5)$$

Then $\omega(e_{i,j})$ is the sum of all the (at most six) values in the set $\tau(e_{i,j} \leftarrow C^*(e_{i,j}))$; each of these values can be any of $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, since $1 \leq |C(e^*)| \leq 6$ for any $e^* \in C^*(e_{i,j})$. We also use the following vectors to represent the *ordered values* of $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot}))$ and $\tau(e_{i,j} \leftarrow C^*(e_{\cdot,j}))$, respectively:

$$\begin{aligned} &(\tau(e_{i,j} \leftarrow e_{i-1,j''-1}^*), \tau(e_{i,j} \leftarrow e_{i,j'}^*), \tau(e_{i,j} \leftarrow e_{i+1,j''' + 1}^*)), \\ &(\tau(e_{i,j} \leftarrow e_{i''-1,j-1}^*), \tau(e_{i,j} \leftarrow e_{i',j}^*), \tau(e_{i,j} \leftarrow e_{i''' + 1,j+1}^*)). \end{aligned}$$

We need the following three more subsets of M , all of which are associated with $e_{i,j} \in M$.

$$\begin{aligned} C(C^*(e_{i,\cdot})) &= \bigcup_{e^* \in C^*(e_{i,\cdot})} C(e^*), \\ C(C^*(e_{\cdot,j})) &= \bigcup_{e^* \in C^*(e_{\cdot,j})} C(e^*), \\ C(C^*(e_{i,j})) &= C(C^*(e_{i,\cdot})) \cup C(C^*(e_{\cdot,j})). \end{aligned}$$

4.2 Value combinations of $\tau(e_{i,j} \leftarrow C^*(e_{i,j}))$ with $\omega(e_{i,j}) \geq 3$

Note that the operation REPLACE-5-BY-6 actually executes swapping p edges of the current compatible matching out for $p+1$ compatible edges to expand the matching, for $p = 1, 2, 3, 4, 5$. Therefore, for any edge $e_{i,j} \in M$, we can never have two edges $e_{i_1,j_1}^*, e_{i_2,j_2}^* \in C^*(e_{i,j})$ such that $|C(e_{i_1,j_1}^*)| = |C(e_{i_2,j_2}^*)| = 1$, that is, both of them conflict with only the edge $e_{i,j}$ in M . Thus we immediately have the following lemma, which has also been observed in [2].

► **Lemma 4.1.** [2] *For any edge $e_{i,j} \in M$, there is at most one edge $e_{i_1,j_1}^* \in C^*(e_{i,j})$ such that $|C(e_{i_1,j_1}^*)| = 1$.*

► **Lemma 4.2.** *For any edge $e_{i,j} \in M$, and for any pair of parallel edges $e_{i_1,j_1}^*, e_{i_1+1,j_1+1}^* \in C^*(e_{i,j})$, $||C(e_{i_1,j_1}^*)| - |C(e_{i_1+1,j_1+1}^*)|| \leq 2$.*

Proof. Since the edges of $C(e_{i_1,j_1}^*) \cup C(e_{i_1+1,j_1+1}^*) \subseteq M$ are pairwise compatible, we have

$$\begin{aligned} C(e_{i_1,j_1}^*) - C(e_{i_1+1,j_1+1}^*) &\subseteq \{e_{i_1-1,\diamond}, e_{\diamond,j_1-1}\}, \\ C(e_{i_1+1,j_1+1}^*) - C(e_{i_1,j_1}^*) &\subseteq \{e_{i_1+2,\diamond}, e_{\diamond,j_1+2}\}, \end{aligned}$$

where $e_{i_1-1,\diamond}$ (e_{\diamond,j_1-1} , $e_{i_1+2,\diamond}$, e_{\diamond,j_1+2} , respectively) denotes the edge of M incident at $d_{i_1-1}^A$ ($d_{j_1-1}^B$, $d_{i_1+2}^A$, $d_{j_1+2}^B$, respectively), if it exists, or otherwise it is a void edge. Thus, $|C(e_{i_1,j_1}^*) - C(e_{i_1+1,j_1+1}^*)| \leq 2$ and $|C(e_{i_1+1,j_1+1}^*) - C(e_{i_1,j_1}^*)| \leq 2$, which together imply $||C(e_{i_1,j_1}^*)| - |C(e_{i_1+1,j_1+1}^*)|| \leq 2$. \blacktriangleleft

► **Lemma 4.3.** *Suppose $|C^*(e_{i,\cdot})| = 3$, then $C^*(e_{i,\cdot}) = \{e_{i-1,j'-1}^*, e_{i,j'}^*, e_{i+1,j'+1}^*\}$ for some $j' \neq j$. In this case we can never have $|C(e_{i-1,j'-1}^*)| = |C(e_{i,j'}^*)| = |C(e_{i+1,j'+1}^*)| = 2$, if one of the following three conditions holds:*

1. *there is an edge $e_{i_1,j_1}^* \in C^*(e_{i,j})$ such that $|C(e_{i_1,j_1}^*)| = 1$;*
2. *$|C(C^*(e_{i,j}))| \leq |C^*(e_{i,j})|$;*
3. *there is at least one singleton edge of M in $C(C^*(e_{i,\cdot}))$.*

Proof. Recall that $C^*(e_{i,\cdot}) = \{e_{i-1,j''-1}^*, e_{i,j'}^*, e_{i+1,j''' +1}^*\}$ for some $j', j'', j''' (\neq j)$. When all these three edges of M^* exist, they are consecutive parallel edges, that is, $j' = j'' = j'''$ and thus $C^*(e_{i,\cdot}) = \{e_{i-1,j'-1}^*, e_{i,j'}^*, e_{i+1,j'+1}^*\}$ for some $j' \neq j$. This proves the first half of the lemma.

Next, assume $|C(e_{i-1,j'-1}^*)| = |C(e_{i,j'}^*)| = |C(e_{i+1,j'+1}^*)| = 2$, and we will show none of the three conditions holds.

Since $e_{i,j} \in C(e_{i-1,j'-1}^*) \cap C(e_{i,j'}^*) \cap C(e_{i+1,j'+1}^*)$, each of $C(e_{i-1,j'-1}^*)$, $C(e_{i,j'}^*)$, $C(e_{i+1,j'+1}^*)$ contains exactly one edge other than $e_{i,j}$. Observe that any edge of M conflicting with $e_{i,j}^*$ must be conflicting with either $e_{i-1,j'-1}^*$ or $e_{i+1,j'+1}^*$. We conclude that either $C(e_{i-1,j'-1}^*) = C(e_{i,j'}^*)$ or $C(e_{i+1,j'+1}^*) = C(e_{i,j'}^*)$, implying that $2 \leq |C(C^*(e_{i,\cdot}))| \leq 3$.

If $|C(C^*(e_{i,\cdot}))| = 2$, then the algorithm \mathcal{LS} would have replaced these two edges of $C(C^*(e_{i,\cdot}))$ by the three edges of $C^*(e_{i,\cdot})$, contradicting to the fact that M is the solution by \mathcal{LS} . Therefore, $|C(C^*(e_{i,\cdot}))| = 3$.

If the first condition holds, the algorithm \mathcal{LS} would have replaced the three edges of $C(C^*(e_{i,\cdot}))$ by the edge e_{i_1,j_1}^* and the three edges of $C^*(e_{i,\cdot})$ to expand M , again a contradiction.

If the second condition holds, we have $|C(C^*(e_{i,j}))| \leq |C(C^*(e_{i,\cdot}))| + |C(C^*(e_{i,j}))| - 1 \leq 2 + |C^*(e_{i,j})| < |C^*(e_{i,j})| \leq 6$. Then, the algorithm \mathcal{LS} would have replaced all the edges of $C(C^*(e_{i,j}))$ by all the edges of $C^*(e_{i,j})$ to expand M , also a contradiction.

When there is at least one singleton edge of M in $C(C^*(e_{i,\cdot}))$, we distinguish two cases where $e_{i,j}$ is singleton or not. If $e_{i,j}$ is not a singleton, then we may assume the edge $e_{i+1,j+1} \in M$ and thus $e_{i+1,j+1} \in C(C^*(e_{i,\cdot}))$ too; it follows from $|C(e_{i-1,j'-1}^*)| = |C(e_{i,j'}^*)| = |C(e_{i+1,j'+1}^*)| = 2$ that these two edges form an isolated pair of parallel edges in M . In this case, the algorithm \mathcal{LS} would have replaced the three edges in $C(C^*(e_{i,\cdot}))$ by the three parallel edges of $C^*(e_{i,\cdot})$ to decrease the number of singleton edges by at least one, a contradiction. If $e_{i,j}$ is a singleton, then the other edge conflicting with $e_{i,j}^*$ must also be a singleton. The algorithm \mathcal{LS} would still have replaced the three edges in $C(C^*(e_{i,\cdot}))$ by the three parallel edges of $C^*(e_{i,\cdot})$ to decrease the number of singleton edges by at least one, again a contradiction.

In summary, we conclude that none of the three conditions would hold. This proves the second half of the lemma. \blacktriangleleft

For an edge $e_{i,j} \in M$ with $\omega(e_{i,j}) \geq 3$, we can now characterize the multi-set $\tau(e_{i,j} \leftarrow C^*(e_{i,j}))$ of six values, in which an entry of 0 represents a void edge in $C^*(e_{i,j})$. We arrange these six values in a non-increasing order. Using the above three Lemmas 4.1–4.3, we have the following conclusion:

► **Lemma 4.4.** *For an edge $e_{i,j} \in M$ with $\omega(e_{i,j}) \geq 3$, there are 8 possible value combinations of $\tau(e_{i,j} \leftarrow C^*(e_{i,j}))$, which are $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}\}$, $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}\}$, $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}\}$, $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$, $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}\}$, $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$, $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$, and $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\}$. These combinations give rise to $\omega(e_{i,j}) = \frac{10}{3}, \frac{13}{4}, \frac{19}{6}, \frac{37}{12}, \frac{91}{30}, 3, 3$ and 3 respectively.*

We remark that in Lemma 4.4, $|C^*(e_{i,j})| = 6$ except for the last combination where $|C^*(e_{i,j})| = 5$. Also, we see that $\max_{e \in M} \omega(e) \leq 10/3$, implying that the algorithm \mathcal{LS} is a $10/3$ -approximation. (This is worse than the current best 3.25 -approximation though.)

4.3 Ordered value combinations of $\tau(e_{i,j} \leftarrow C^*(e_{i.,}))$ with $\omega(e_{i,j}) \geq 3$

We discuss the possible ordered value combinations of $\tau(e_{i,j} \leftarrow C^*(e_{i.,}))$ in this section.

Using the first condition of Lemma 4.3, we can rule out $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$ for $\tau(e_{i,j} \leftarrow C^*(e_{i.,}))$. From the 8 possible value combinations of $\tau(e_{i,j} \leftarrow C^*(e_{i,j}))$ stated in Lemma 4.4, by Lemma 4.2 we can identify in total 12 possible value combinations of $\tau(e_{i,j} \leftarrow C^*(e_{i.,}))$ with $\omega(e_{i,j}) \geq 3$, stated in the following lemma.

► **Lemma 4.5.** *For an edge $e_{i,j} \in M$ with $\omega(e_{i,j}) \geq 3$, there are 12 possible value combinations of $\tau(e_{i,j} \leftarrow C^*(e_{i.,}))$, which are $\{1, \frac{1}{2}, \frac{1}{2}\}$, $\{1, \frac{1}{2}, \frac{1}{3}\}$, $\{1, \frac{1}{2}, \frac{1}{4}\}$, $\{1, \frac{1}{3}, \frac{1}{3}\}$, $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{3}\}$, $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{4}\}$, $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{3}\}$, $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$, $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\}$, $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$, $\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$, and $\{\frac{1}{2}, \frac{1}{2}, 0\}$.*

► **Lemma 4.6.** *Suppose $|C^*(e_{i.,})| = 3$ and $C^*(e_{i.,}) = \{e_{i-1,j'-1}^*, e_{i,j'}^*, e_{i+1,j'+1}^*\}$ for some $j' \neq j$. We have*

$$C(e_{i,j'}^*) \subseteq C(e_{i-1,j'-1}^*) \cup C(e_{i+1,j'+1}^*), \quad (6)$$

$$|C(C^*(e_{i.,}))| \leq |C(e_{i-1,j'-1}^*)| + |C(e_{i+1,j'+1}^*)| - 1, \quad (7)$$

$$|C(C^*(e_{i.,}))| \geq \max \begin{cases} 3, \\ |C(e_{i-1,j'-1}^*)| + |C(e_{i+1,j'+1}^*)| - 2, \\ |C(e_{i-1,j'-1}^*)| + |C(e_{i+1,j'+1}^*)| - |C(e_{i,j'}^*)|. \end{cases} \quad (8)$$

Proof. Observe that any edge of M conflicting with $e_{i,j'}^*$ must also conflict with either $e_{i-1,j'-1}^*$ or $e_{i+1,j'+1}^*$. We have $C(e_{i,j'}^*) \subseteq C(e_{i-1,j'-1}^*) \cup C(e_{i+1,j'+1}^*)$, which proves the inequality (6) and also indicates that $C(C^*(e_{i.,})) = C(e_{i-1,j'-1}^*) \cup C(e_{i+1,j'+1}^*)$. Since $e_{i,j} \in C(e_{i-1,j'-1}^*) \cap C(e_{i,j'}^*) \cap C(e_{i+1,j'+1}^*) \subseteq \{e_{i,j}, e_{\diamond,j'}\}$, where $e_{\diamond,j'}$ is a possible edge of M incident at $d_{j'}^B$, we have

$$|C(e_{i-1,j'-1}^*)| + |C(e_{i+1,j'+1}^*)| - 2 \leq |C(C^*(e_{i.,}))| \leq |C(e_{i-1,j'-1}^*)| + |C(e_{i+1,j'+1}^*)| - 1.$$

This proves the inequality (7) and the second inequality in (8).

Also observe that any edge of M conflicting with both $e_{i-1,j'-1}^*$ and $e_{i+1,j'+1}^*$ must conflict with $e_{i,j'}^*$ too. We have $C(e_{i-1,j'-1}^*) \cap C(e_{i+1,j'+1}^*) \subseteq C(e_{i,j'}^*)$. Therefore,

$$|C(C^*(e_{i.,}))| \geq |C(e_{i-1,j'-1}^*)| + |C(e_{i+1,j'+1}^*)| - |C(e_{i,j'}^*)|,$$

proving the last inequality in (8). $|C(C^*(e_{i.,}))| \geq 3$ can be proven by a simple contradiction, since otherwise the algorithm \mathcal{LS} would replace all the edges of $C(C^*(e_{i.,}))$ by the three edges of $C^*(e_{i.,})$ to expand M . ◀

► **Lemma 4.7.** *Suppose $|C^*(e_{i.,})| = 3$ and $C^*(e_{i.,}) = \{e_{i-1,j'-1}^*, e_{i,j'}^*, e_{i+1,j'+1}^*\}$ for some $j' \neq j$, and there is an edge $e_{i_3,j_3}^* \in C^*(e_{i.,})$ such that $|C(e_{i_3,j_3}^*)| = 1$. For any two edges $e_{i_1,j_1}^*, e_{i_2,j_2}^* \in C^*(e_{i.,})$, if $|C(e_{i_1,j_1}^*)| = |C(e_{i_2,j_2}^*)| = 2$, then the following two statements hold:*

1. e_{i_1, j_1}^* and e_{i_2, j_2}^* are parallel, that is, either $i_2 = i_1 + 1, j_2 = j_1 + 1$ or $i_2 = i_1 - 1, j_2 = j_1 - 1$.
2. $C(e_{i_1, j_1}^*) \cap C(e_{i_2, j_2}^*) = \{e_{i, j}\}$.

Proof. Using $|C(e_{i_3, j_3}^*)| = 1$, we know from Lemma 4.1 that $|C(e_{i-1, j'-1}^*)| \geq 2$, $|C(e_{i, j'}^*)| \geq 2$ and $|C(e_{i+1, j'+1}^*)| \geq 2$.

To prove the first statement, we suppose to the contrary that $i_1 = i - 1$ and $i_2 = i + 1$, and thus $|C(e_{i-1, j'-1}^*)| = |C(e_{i+1, j'+1}^*)| = 2$. From the inequality (7) of Lemma 4.6, we have $|C(e_{i, j'}^*)| \leq |C(C^*(e_{i, \cdot}))| \leq 3$. Since Lemma 4.3 has ruled out the possibility of $|C(e_{i, j'}^*)| = 2$, we have $|C(e_{i, j'}^*)| = |C(C^*(e_{i, \cdot}))| = 3$. However, the algorithm \mathcal{LS} would replace the three edges of $C(C^*(e_{i, \cdot}))$ by e_{i_3, j_3}^* and the three edges of $C^*(e_{i, \cdot})$ to expand M , a contradiction.

Based on the first statement, we assume without loss of generality that $|C(e_{i, j'}^*)| = |C(e_{i-1, j'-1}^*)| = 2$. Note that $e_{i, j} \in C(e_{i, j'}^*) \cap C(e_{i-1, j'-1}^*)$. If $C(e_{i, j'}^*) = C(e_{i-1, j'-1}^*)$, then the algorithm \mathcal{LS} would replace the two edges of $C(e_{i, j'}^*)$ by the three edges $e_{i, j'}^*$, $e_{i-1, j'-1}^*$, e_{i_3, j_3}^* to expand M , a contradiction. Therefore, $C(e_{i, j'}^*) \neq C(e_{i-1, j'-1}^*)$, which implies the second statement $C(e_{i_1, j_1}^*) \cap C(e_{i_2, j_2}^*) = \{e_{i, j}\}$. ◀

Note that each value combination $\{\tau_1, \tau_2, \tau_3\}$ of $\tau(e_{i, j} \leftarrow C^*(e_{i, \cdot}))$ in Lemma 4.5 gives rise to $3! = 6$ different ordered value combinations. Due to symmetry, we consider only three of them: (τ_2, τ_1, τ_3) , (τ_1, τ_2, τ_3) , and (τ_1, τ_3, τ_2) , in the following to determine whether or not they can be possible ordered value combinations for $\tau(e_{i, j} \leftarrow C^*(e_{i, \cdot}))$.

1. $\tau(e_{i, j} \leftarrow C^*(e_{i, \cdot})) = \{1, \frac{1}{2}, \frac{1}{2}\}$.
The case of $(1, \frac{1}{2}, \frac{1}{2})$ can be ruled out by the inequalities (7) and (8) of Lemma 4.6.
Then the only possible case left is $(\frac{1}{2}, 1, \frac{1}{2})$.
2. $\tau(e_{i, j} \leftarrow C^*(e_{i, \cdot})) = \{1, \frac{1}{2}, \frac{1}{3}\}$.
The case of $(1, \frac{1}{3}, \frac{1}{2})$ can immediately be ruled out by the inequality (7) of Lemma 4.6.
Then the two possible cases left are $(\frac{1}{2}, 1, \frac{1}{3})$ and $(1, \frac{1}{2}, \frac{1}{3})$.
3. $\tau(e_{i, j} \leftarrow C^*(e_{i, \cdot})) = \{1, \frac{1}{2}, \frac{1}{4}\}$.
Both cases of $(\frac{1}{2}, 1, \frac{1}{4})$ and $(1, \frac{1}{4}, \frac{1}{2})$ can immediately be ruled out by Lemma 4.2.
Then the only possible case left is $(1, \frac{1}{2}, \frac{1}{4})$.
4. $\tau(e_{i, j} \leftarrow C^*(e_{i, \cdot})) = \{1, \frac{1}{3}, \frac{1}{3}\}$.
Consider the case of $\tau(e_{i, j} \leftarrow C^*(e_{i, \cdot})) = (1, \frac{1}{3}, \frac{1}{3})$. In this case, we have $C(C^*(e_{i, \cdot})) = C(e_{i, j'}^*) = C(e_{i+1, j'+1}^*)$ with $|C(C^*(e_{i, \cdot}))| = 3$, indicating that one of the three edges in $C(C^*(e_{i, \cdot}))$ must be a singleton edge of M and there is no edge in $M - C(C^*(e_{i, \cdot}))$ parallel with any edge in $C(C^*(e_{i, \cdot}))$. However, the algorithm \mathcal{LS} would replace the three edges of $C(C^*(e_{i, \cdot}))$ by the three parallel edges of $C^*(e_{i, \cdot})$ to reduce the singleton edges in M , a contradiction.
Thus the only possible case left is $(\frac{1}{3}, 1, \frac{1}{3})$.
5. $\tau(e_{i, j} \leftarrow C^*(e_{i, \cdot})) = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{3}\}$.
The case of $(\frac{1}{2}, \frac{1}{3}, \frac{1}{2})$ can immediately be ruled out by Lemma 4.7.
Then the only possible case left is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{3})$.
6. $\tau(C^*(e_{i, \cdot})) = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{4}\}$.
The case of $(\frac{1}{2}, \frac{1}{4}, \frac{1}{2})$ can immediately be ruled out by the inequality (7) of Lemma 4.6.
Then the only possible case left is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{4})$.
7. $\tau(e_{i, j} \leftarrow C^*(e_{i, \cdot})) = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{3}\}$.
Both cases of $(\frac{1}{3}, \frac{1}{2}, \frac{1}{3})$ and $(\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$ are possible.
8. $\tau(e_{i, j} \leftarrow C^*(e_{i, \cdot})) = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$.
All three cases of $(\frac{1}{3}, \frac{1}{2}, \frac{1}{4})$, $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$, and $(\frac{1}{2}, \frac{1}{4}, \frac{1}{3})$ are possible.

9. $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\}$.
Both cases of $(\frac{1}{3}, \frac{1}{2}, \frac{1}{5})$ and $(\frac{1}{2}, \frac{1}{5}, \frac{1}{3})$ can immediately be ruled out by Lemma 4.2.
Then the only possible case left is $(\frac{1}{2}, \frac{1}{3}, \frac{1}{5})$.
10. $\tau(C^*(e_{i,\cdot})) = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$.
Both cases of $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ are possible.
11. $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$.
The only case $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is possible.
12. $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) = \{\frac{1}{2}, \frac{1}{2}, 0\}$.
Both cases of $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{2}, 0, \frac{1}{2})$ are possible.

We summarize the above discussion in the following lemma:

► **Lemma 4.8.** *For an edge $e_{i,j} \in M$ with $\omega(e_{i,j}) \geq 3$, there are 18 possible ordered value combinations of $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot}))$, which are $(\frac{1}{2}, 1, \frac{1}{2})$, $(\frac{1}{2}, 1, \frac{1}{3})$, $(1, \frac{1}{2}, \frac{1}{3})$, $(1, \frac{1}{2}, \frac{1}{4})$, $(\frac{1}{3}, 1, \frac{1}{3})$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{3})$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{4})$, $(\frac{1}{3}, \frac{1}{2}, \frac{1}{3})$, $(\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$, $(\frac{1}{3}, \frac{1}{2}, \frac{1}{4})$, $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$, $(\frac{1}{2}, \frac{1}{4}, \frac{1}{3})$, $(\frac{1}{2}, \frac{1}{3}, \frac{1}{5})$, $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $(\frac{1}{2}, \frac{1}{2}, 0)$, and $(\frac{1}{2}, 0, \frac{1}{2})$.*

4.4 Edge combinations of $C(C^*(e_{i,j}))$ with $\omega(e_{i,j}) \geq 3$

We examine all possible combinations of the edges in $C(C^*(e_{i,\cdot}))$ with $\omega(e_{i,j}) \geq 3$. We distinguish two cases where $e_{i,j} \in p(M)$ and $e_{i,j} \in s(M)$, respectively. In fact, as shown in Section 4.4.1, the edge $e_{i,j}$ cannot be a parallel edge in M .

4.4.1 $e_{i,j}$ cannot be a parallel edge of M

Recall that the number of singleton edges of the maximal compatible matching M cannot be further reduced by the algorithm \mathcal{LS} using the operation REDUCE-5-BY-5.

We assume to the contrary that $e_{i,j} \in p(M)$, and assume that $e_{i+1,j+1} \in p(M)$ too.

From $|C^*(e_{i,j})| \geq 5$ in Lemma 4.4, we consider $|C^*(e_{i,\cdot})| = 3$ and suppose that $C^*(e_{i,\cdot}) = \{e_{i-1,j'-1}^*, e_{i,j'}^*, e_{i+1,j'+1}^*\}$ for some $j' \neq j$.

Clearly, $|C(e_{i,j'}^*)| \geq 2$ and $|C(e_{i+1,j'+1}^*)| \geq 2$ since both contain the edges $e_{i,j}$ and $e_{i+1,j+1}$. It follows that the middle value in the ordered value combination of $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot}))$ must be $\leq \frac{1}{2}$. This rules out three of the 18 possible ordered value combinations stated in Lemma 4.8, each having a 1 in the middle, which are $(\frac{1}{2}, 1, \frac{1}{2})$, $(\frac{1}{2}, 1, \frac{1}{3})$, $(\frac{1}{3}, 1, \frac{1}{3})$. Furthermore, since $(\frac{1}{2}, 1, \frac{1}{2})$ is the only one resulted from the (unordered) value combination $\{1, \frac{1}{2}, \frac{1}{2}\}$, we conclude that it is impossible to have $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) = \{1, \frac{1}{2}, \frac{1}{2}\}$. For the same reason, it is impossible to have $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) = \{1, \frac{1}{3}, \frac{1}{3}\}$.

When $|C^*(e_{\cdot,j})| = 3$, the argument in the last paragraph applies to $C^*(e_{\cdot,j})$ too.

Consider the 8 possible value combinations of $\tau(e_{i,j} \leftarrow C^*(e_{i,j}))$ such that $\omega(e_{i,j}) \geq 3$, in Lemma 4.4. We observe that $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) \in \{\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}, \{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\}, \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}, \{\frac{1}{2}, \frac{1}{2}, 0\}\}$ only if $\tau(e_{i,j} \leftarrow C^*(e_{\cdot,j})) = \{1, \frac{1}{2}, \frac{1}{2}\}$, which is impossible to happen. We thus conclude that only 5 out of the 12 value combinations of $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot}))$ in Lemma 4.5 remain possible, which are $\{1, \frac{1}{2}, \frac{1}{3}\}$, $\{1, \frac{1}{2}, \frac{1}{4}\}$, $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{3}\}$, $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{4}\}$, and $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{3}\}$. These give 6 possible ordered value combinations of $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot}))$, which are, $(1, \frac{1}{2}, \frac{1}{3})$, $(1, \frac{1}{2}, \frac{1}{4})$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{3})$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{4})$, $(\frac{1}{3}, \frac{1}{2}, \frac{1}{3})$, and $(\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$.

In the rest of this section, we have $|C^*(e_{i,j})| = 6$, $C^*(e_{i,\cdot}) = \{e_{i-1,j'-1}^*, e_{i,j'}^*, e_{i+1,j'+1}^*\}$ for some $j' \neq j$, and $C^*(e_{\cdot,j}) = \{e_{i'-1,j-1}^*, e_{i',j}^*, e_{i'+1,j+1}^*\}$ for some $i' \neq i$.

► **Lemma 4.9.** *For the pair of parallel edges $e_{i,j}, e_{i+1,j+1} \in p(M)$, $C^*(e_{i,j}) \cap C^*(e_{i+1,j+1}) = \{e_{i,j'}^*, e_{i+1,j'+1}^*, e_{i',j}^*, e_{i'+1,j+1}^*\}$. If $|C(e_{i-1,j'-1}^*)| = 1$, then there is at most one edge $e_{i_1,j_1}^* \in C^*(e_{i,j}) \cap C^*(e_{i+1,j+1})$ such that $|C(e_{i_1,j_1}^*)| = 2$.*

Proof. The first half of the lemma is trivial. For the second half, we note that $C(e_{i-1,j'-1}^*) = \{e_{i,j}\}$; if there is another edge $e_{i_2,j_2}^* \in C^*(e_{i,j}) \cap C^*(e_{i+1,j+1})$ such that $|C(e_{i_2,j_2}^*)| = 2$, that is, $C(e_{i_1,j_1}^*) = C(e_{i_2,j_2}^*) = \{e_{i,j}, e_{i+1,j+1}\}$, then the algorithm \mathcal{LS} would replace the two edges $e_{i,j}$ and $e_{i+1,j+1}$ by the three edges $e_{i-1,j'-1}^*$, e_{i_1,j_1}^* , e_{i_2,j_2}^* to expand M , a contradiction. \blacktriangleleft

Lemma 4.9 states that when $e_{i,j}$ is a parallel edge of M , the value combination of $\tau(e_{i,j} \leftarrow C^*(e_{i,j}))$ contains at most two $\frac{1}{2}$'s, besides the value 1. Among the 8 possible value combinations of $\tau(e_{i,j} \leftarrow C^*(e_{i,j}))$ such that $\omega(e_{i,j}) \geq 3$, in Lemma 4.4, the only one with two $\frac{1}{2}$'s is $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$. This leaves only two possible ordered value combinations of $\tau(e_{i,j} \leftarrow C^*(e_{i,j}))$, which are, $(1, \frac{1}{2}, \frac{1}{3})$ and $(\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$.

Assume $\tau(e_{i,j} \leftarrow C^*(e_{i,j})) = (1, \frac{1}{2}, \frac{1}{3})$ and $\tau(e_{i,j} \leftarrow C^*(e_{i,j})) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$ (or the other way around). By the inequalities (7) and (8) in Lemma 4.6, we have $|C(C^*(e_{i,j}))| = 3$ and $3 \leq |C(C^*(e_{i,j}))| \leq 4$. Since $\{e_{i,j}, e_{i+1,j+1}\} \subseteq C(C^*(e_{i,j})) \cap C(C^*(e_{i,j}))$, we have $4 \leq |C(C^*(e_{i,j}))| \leq 5$. Thus, the algorithm \mathcal{LS} would replace all the edges of $C(C^*(e_{i,j}))$ by the six edges of $C^*(e_{i,j})$ to expand M . This contradiction leaves no ordered value combination of $\tau(e_{i,j} \leftarrow C^*(e_{i,j}))$. We thus have proved the following lemma:

► Lemma 4.10. *When the edge $e_{i,j}$ is a parallel edge of M , there is no value combination of $\tau(e_{i,j} \leftarrow C^*(e_{i,j}))$ such that $\omega(e_{i,j}) \geq 3$.*

4.4.2 $e_{i,j}$ is a singleton edge of M

With $e_{i,j} \in s(M)$, we discuss each of the 18 possible ordered value combinations of $\tau(e_{i,j} \leftarrow C^*(e_{i,j}))$ listed in Lemma 4.8.

Consider an edge $e_{h,\ell} \in C(C^*(e_{i,j}))$, $e_{h,\ell} \neq e_{i,j}$. Note that $e_{h,\ell}$ might be parallel with an edge in $M - C(C^*(e_{i,j}))$. We define $\mathcal{N}_p(e_{h,\ell})$ to be the subset of the maximal consecutive parallel (to $e_{h,\ell}$) edges in $M - C(C^*(e_{i,j}))$. Therefore, $\mathcal{N}_p(e_{h,\ell})$ will be either $\{e_{h+1,\ell+1}, \dots, e_{h+q,\ell+q}\}$ or $\{e_{h-1,\ell-1}, \dots, e_{h-q,\ell-q}\}$, for some $q \geq 0$ (when $q = 0$, this set is empty). Let

$$\mathcal{N}_p(C(C^*(e_{i,j}))) = \bigcup_{e_{h,\ell} \in C(C^*(e_{i,j}))} \mathcal{N}_p(e_{h,\ell}),$$

and

$$\mathcal{N}_p[C(C^*(e_{i,j}))] = \mathcal{N}_p(C(C^*(e_{i,j}))) \cup C(C^*(e_{i,j})).$$

Recall that, in general, the subscript of a vertex of D^A has an i or h , and the subscript of a vertex of D^B has a j or ℓ . In the sequel, for simplicity, we use e_h and e_ℓ (e_h^* and e_ℓ^* , respectively) to denote the edges of M (M^* , respectively) incident at the vertices d_i^A and d_j^B , respectively, if they exist, or otherwise they are void edges.

We next discuss all possible configurations of the edges of $C^*(e_{i,j})$ and $C(C^*(e_{i,j}))$ in figures, associated with each of the 18 ordered value combinations of $\tau(e_{i,j} \leftarrow C^*(e_{i,j}))$ listed in Lemma 4.8. We adopt the following scheme for graphically presenting a configuration: In each figure (for example, Fig. 4.1), the edge $e_{i,j}$ is in the bold solid line; the edges in vertical bold dashed lines are in $C^*(e_{i,j})$ (for example, $e_{i,j'}^*$); the edges in thin solid lines are edges in $C(C^*(e_{i,j}))$ (for example, e_{i+2}); and the edges in thin dashed lines are edges in $\mathcal{N}_p(C(C^*(e_{i,j})))$ (for example, e_{i+3}); the vertices in filled circles are surely not incident with any edge of M (for example, $i - 2$); the vertices in hollow circles have uncertain incidence in M (for example, $j' - 2$).

We remind the readers that if there is no entry 1 in a value combination of $\tau(e_{i,j} \leftarrow C^*(e_{i,.}))$, then there must be an entry 1 in the corresponding value combination of $\tau(e_{i,j} \leftarrow C^*(e_{.,j}))$, that is, there is an edge $e_{i_1,j_1}^* \in C^*(e_{.,j})$ such that $|C(e_{i_1,j_1}^*)| = 1$.

1. $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (\frac{1}{2}, 1, \frac{1}{2})$: According to the inequalities (7) and (8) of Lemma 4.6, we have $|C(C^*(e_{i,.}))| = 3$. There is exactly one edge of M incident at either of $i+2$ and $j'+2$ but not both. We assume $e_{i+2} \in M$. If e_{i+2} is a singleton edge of M or $|\mathcal{N}_p(e_{i+2})| \geq 2$, then the algorithm \mathcal{LS} would replace $e_{i,j}$ and e_{i+2} by the two parallel edges $e_{i,j'}^*$ and $e_{i+1,j'+1}^*$ to reduce the singleton edges, a contradiction. Therefore, we have $e_{i+3} \in M$ but no edge of M is incident at $i+4$. The incidence at $i-2$ and $j'-2$ and further to the left can be symmetrically discussed. In this sense, there is only one possible edge combination of $C(C^*(e_{i,.}))$, as shown in Fig. 4.1 with $e_{i+2}, e_{j'-2} \in M$, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ is also shown.

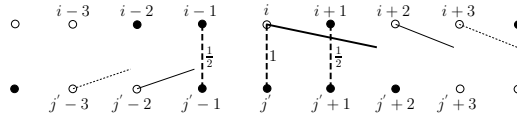


Figure 4.1 The only possible configuration of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ when $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (\frac{1}{2}, 1, \frac{1}{2})$. We have $|C(C^*(e_{i,.}))| = 3$, and $|\mathcal{N}_p(e_{i+2})| = |\mathcal{N}_p(e_{j'-2})| = 1$ in this configuration. It also represents the other three symmetric configurations where $|\mathcal{N}_p(e_{i+2})| = |\mathcal{N}_p(e_{i-2})| = 1$, $|\mathcal{N}_p(e_{j'+2})| = |\mathcal{N}_p(e_{j'-2})| = 1$, and $|\mathcal{N}_p(e_{j'+2})| = |\mathcal{N}_p(e_{i-2})| = 1$, respectively. (Recall that the edge $e_{i,j}$ is in bold solid line, the edges in vertical bold dashed lines are in $C^*(e_{i,.})$, the edges in thin solid lines are in $C(C^*(e_{i,.}))$, and the edges in thin dashed lines are in $\mathcal{N}_p[C(C^*(e_{i,.}))]$; the vertices in filled circles are surely incident with no edges of M and the vertices in hollow circles have uncertain incidence in M .)

2. $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (\frac{1}{2}, 1, \frac{1}{3})$: We have $C(e_{i,j'}^*) \subset C(e_{i-1,j'-1}^*)$, and thus $|C(e_{i,j'}^*) \cup C(e_{i-1,j'-1}^*)| = 2$ and $|C(C^*(e_{i,.}))| = 4$. There is exactly one edge of M incident at either of $i-2$ and $j'-2$ but not both. We assume $e_{j'-2} \in M$. If $e_{j'-2}$ is a singleton edge of M or $|\mathcal{N}_p(e_{j'-2})| \geq 2$, then the algorithm \mathcal{LS} would replace $e_{i,j}$ and $e_{j'-2}$ by the two parallel edges $e_{i,j'}^*$ and $e_{i-1,j'-1}^*$ to reduce the singleton edges, a contradiction. Therefore, we have $e_{j'-3} \in M$ but no edge of M is incident at $j'-4$. In this sense, there is only one possible edge combination of $C(C^*(e_{i,.}))$, as shown in Fig. 4.2 with $e_{j'-2} \in M$, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ is also shown.

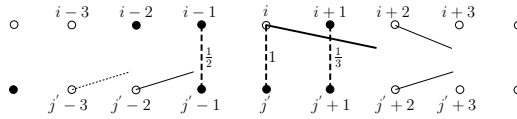
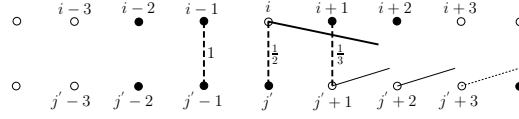


Figure 4.2 The only possible configuration of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ when $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (\frac{1}{2}, 1, \frac{1}{3})$. We have $|C(C^*(e_{i,.}))| = 4$ and $|\mathcal{N}_p(e_{j'-2})| = 1$ in this configuration. It also represents the symmetric configuration where $|\mathcal{N}_p(e_{i-2})| = 1$.

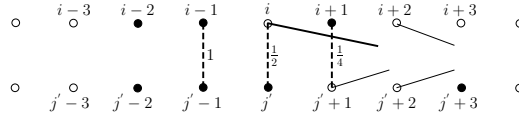
3. $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (1, \frac{1}{2}, \frac{1}{3})$: According to the inequalities (7) and (8) of Lemma 4.6, we have $|C(C^*(e_{i,.}))| = 3$. Since $e_{i,j}$ is a singleton edge of M , $e_{j'+1} \in M$; and either $e_{i+2} \in M$ or $e_{j'+2} \in M$ but not both. If $e_{i+2} \in M$, then $e_{j'+1}$ is a singleton edge of M , and thus the algorithm \mathcal{LS} would replace $e_{i,j}$ and $e_{j'+1}$ by the two parallel edges $e_{i,j'}^*$ and $e_{i-1,j'-1}^*$ to reduce the singleton edges, a contradiction. Therefore, $e_{j'+2} \in M$. Similarly, if $\mathcal{N}_p(e_{j'+2}) = \emptyset$ or $|\mathcal{N}_p(e_{j'+2})| \geq 2$, then the algorithm \mathcal{LS} would replace the three edges

of $C(C^*(e_{i,\cdot}))$ by the three parallel edges of $C^*(e_{i,\cdot})$ to reduce the singleton edges, a contradiction. This leaves the only possible configuration with $|\mathcal{N}_p(e_{j'+2})| = 1$, as shown in Fig. 4.3, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,\cdot}))]$ is also shown.



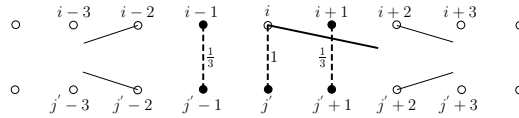
■ **Figure 4.3** The only possible configuration of $\mathcal{N}_p[C(C^*(e_{i,\cdot}))]$ when $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) = (1, \frac{1}{2}, \frac{1}{3})$. We have $|C(C^*(e_{i,\cdot}))| = 3$ and $|\mathcal{N}_p(e_{j'+2})| = 1$ in this configuration.

4. $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) = (1, \frac{1}{2}, \frac{1}{4})$: We have $e_{j'} \notin M$ and $|C(C^*(e_{i,\cdot}))| = 4$. Therefore, $e_{i+2}, e_{j'+1}, e_{j'+2} \in M$. If $|\mathcal{N}_p(e_{j'+2})| \geq 1$, then the algorithm \mathcal{LS} would replace $e_{i,j}$ and $e_{j'+1}$ by the two parallel edges $e_{i,j'}^*$ and $e_{i-1,j'-1}^*$ to reduce the singleton edges, a contradiction. Therefore, $\mathcal{N}_p(e_{j'+2}) = \emptyset$, that is, $e_{j'+3} \notin M$. There is only one possible edge combination of $C(C^*(e_{i,\cdot}))$, as shown in Fig. 4.4, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,\cdot}))]$ is also shown.



■ **Figure 4.4** The only possible configuration of $\mathcal{N}_p[C(C^*(e_{i,\cdot}))]$ when $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) = (1, \frac{1}{2}, \frac{1}{4})$. We have $|C(C^*(e_{i,\cdot}))| = 4$ and $\mathcal{N}_p(e_{j'+2}) = \emptyset$ in this configuration.

5. $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) = (\frac{1}{3}, 1, \frac{1}{3})$: According to the inequalities (7) and (8) of Lemma 4.6, we have $|C(C^*(e_{i,\cdot}))| = 5$. There is only one possible edge combination of $C(C^*(e_{i,\cdot}))$, which is shown in Fig. 4.5, where any configuration of $\mathcal{N}_p[C(C^*(e_{i,\cdot}))]$ is possible.



■ **Figure 4.5** The only possible configuration of $\mathcal{N}_p[C(C^*(e_{i,\cdot}))]$ when $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) = (\frac{1}{3}, 1, \frac{1}{3})$, where $|C(C^*(e_{i,\cdot}))| = 5$.

6. $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{3})$: According to Lemma 4.7, we have $C(e_{i,j'}^*) \cap C(e_{i-1,j'-1}^*) = \{e_{i,j}\}$; thus $e_{j'+1} \in M$, either $e_{i'-2} \in M$ or $e_{j'-2} \in M$ but no both, either $e_{i+2} \in M$ or $e_{j'+2} \in M$ but no both, and $|C(C^*(e_{i,\cdot}))| = 4$. We assume $e_{j'-2} \in M$ ($e_{i'-2} \in M$ is discussed the same). When $e_{i+2} \in M$, $e_{j'+1}$ is a singleton edge of M . If $e_{j'-2}$ is also a singleton edge of M , then the algorithm \mathcal{LS} would replace the four edges in $C(C^*(e_{i,\cdot}))$ by the three parallel edges in $C^*(e_{i,\cdot})$ and $e_{i-1,j'-1}^*$ to reduce the singleton edges, a contradiction. Therefore in this case we have $|\mathcal{N}_p(e_{j'-2})| \geq 1$, that is, $e_{j'-3} \in M$. Similarly, if e_{i+2} is a singleton edge of M or $|\mathcal{N}_p(e_{i+2})| \geq 2$, then the algorithm \mathcal{LS} would replace the three edges $e_{i,j}, e_{j'+1}, e_{i+2}$ by the two parallel edges $e_{i,j'}^*, e_{i+1,j'+1}^*$ and $e_{i-1,j'-1}^*$ to reduce the singleton edges, a contradiction. That is, $e_{i+3} \in M$ but $e_{i+4} \notin M$. This edge combination of $C(C^*(e_{i,\cdot}))$ is shown in Fig. 4.6b, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,\cdot}))]$ is also shown.

When $e_{j'+2} \in M$, for the same reason, if $|\mathcal{N}_p(e_{j'+2})| \neq 1$ then $e_{j'-2}$ must not be a singleton edge of M . This edge combination of $C(C^*(e_{i.,.}))$ is shown in Fig. 4.6a, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i.,.}))]$ is also shown.

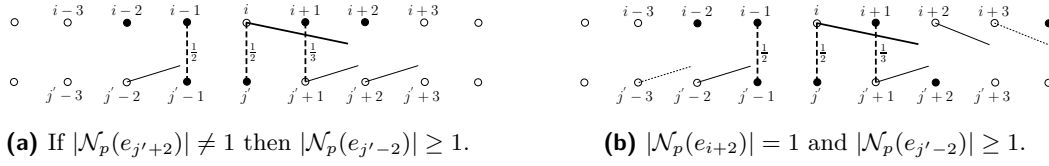


Figure 4.6 The two possible configurations of $\mathcal{N}_p[C(C^*(e_{i.,.}))]$ when $\tau(e_{i,j} \leftarrow C^*(e_{i.,.})) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{3})$. We have $|C(C^*(e_{i.,.}))| = 4$. They also represent the symmetric case where $e_{i'-2} \in M$ instead of $e_{j'-2} \in M$.

7. $\tau(e_{i,j} \leftarrow C^*(e_{i.,.})) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{4})$: According to Lemma 4.7, we have $C(e_{i,j}') \cap C(e_{i-1,j'-1}') = \{e_{i,j}\}$; thus $e_{j'+1} \in M$, either $e_{i'-2} \in M$ or $e_{j'-2} \in M$ but no both, $e_{i+2}, e_{j'+2} \in M$, and $|C(C^*(e_{i.,.}))| = 5$. We assume $e_{j'-2} \in M$ ($e_{i'-2} \in M$ is discussed the same). If $e_{j'-2}$ is a singleton edge of M and $|\mathcal{N}_p(e_{j'+2})| \geq 1$, then the algorithm \mathcal{LS} would replace the three edges $e_{i,j}$, $e_{j'-2}$, and $e_{j'+1}$ by e_{i,j_1}^* and the two parallel edges $e_{i,j'}^*$ and $e_{i-1,j'-1}'^*$ to reduce the singleton edges, a contradiction. Therefore, $|\mathcal{N}_p(e_{j'-2})| \geq 1$ (shown in Fig. 4.7b) or $\mathcal{N}_p(e_{j'+2}) = \emptyset$ (shown in Fig. 4.7a). These two edge combinations of $C(C^*(e_{i.,.}))$ are shown in Fig. 4.7a and Fig. 4.7b, respectively, where the corresponding configurations of $\mathcal{N}_p[C(C^*(e_{i.,.}))]$ are also shown.

Between the two configurations shown in Fig. 4.7a and Fig. 4.7b, we notice that for every edge $e \in C(C^*(e_{i.,.})) - \{e_{i,j}\}$, the largest possible value for $\omega(e)$ in Fig. 4.7a is at least as large as in Fig. 4.7b. Since we are interested in the worst-case analysis, we say Fig. 4.7b is *shadowed* by Fig. 4.7a and we will consider Fig. 4.7a only in the sequel.

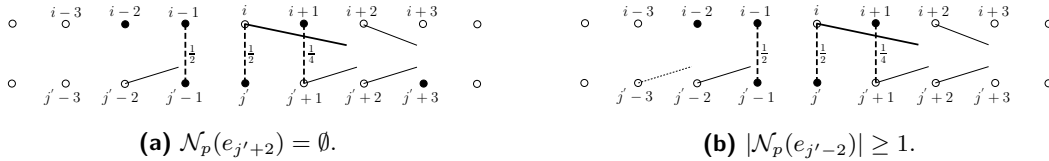


Figure 4.7 The two possible configurations of $\mathcal{N}_p[C(C^*(e_{i.,.}))]$ when $\tau(e_{i,j} \leftarrow C^*(e_{i.,.})) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{4})$. They are associated with the only possible edge combination of $C(C^*(e_{i.,.}))$ with $|C(C^*(e_{i.,.}))| = 5$, which also represents the symmetric case where $e_{i'-2} \in M$ instead of $e_{j'-2} \in M$. The first configuration shadows the second one.

8. $\tau(e_{i,j} \leftarrow C^*(e_{i.,.})) = (\frac{1}{3}, \frac{1}{2}, \frac{1}{3})$: According to the inequalities (7) and (8) of Lemma 4.6, we have $4 \leq |C(C^*(e_{i.,.}))| \leq 5$. Since $i-1$ and $i+1$ are symmetric with respect to i , we only discuss one of them. We have either $e_{j'} \in M$ or $e_{j'-1} \in M$, but not both.

When $e_{j'} \in M$, then either $e_{i-2} \in M$ or $e_{j'-2} \in M$, but not both. We assume $e_{j'-2} \in M$. Similarly, either $e_{i+2} \in M$ or $e_{j'+2} \in M$, but not both. We assume $e_{i+2} \in M$. If e_{i+2} is a singleton edge of M or $|\mathcal{N}_p(e_{i+2})| \geq 2$, then the algorithm \mathcal{LS} would replace the three edges $e_{i,j}$, $e_{j'}$, e_{i+2} by e_{i,j_1}^* and the two parallel edges $e_{i,j'}^*$ and $e_{i+1,j'+1}'^*$ to reduce the singleton edges, a contradiction. Therefore, $|\mathcal{N}_p(e_{i+2})| = 1$; for the same reason, $|\mathcal{N}_p(e_{j'-2})| = 1$. This edge combination of $C(C^*(e_{i.,.}))$ is shown in Fig. 4.8a, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i.,.}))]$ is also shown.

When $e_{j'-1} \in M$, then still either $e_{i-2} \in M$ or $e_{j'-2} \in M$, but not both. On the other side, $e_{i+2} \in M$ and $e_{j'+2} \in M$. When $e_{i-2} \in M$, $e_{j'-1}$ is a singleton edge of M ; and

therefore $|\mathcal{N}_p(e_{i'-2})| = 1$. This edge combination of $C(C^*(e_{i,\cdot}))$ is shown in Fig. 4.8b, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,\cdot}))]$ is also shown.

When $e_{j'-2} \in M$, the edge combination of $C(C^*(e_{i,\cdot}))$ is shown in Fig. 4.8c, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,\cdot}))]$ is also shown.

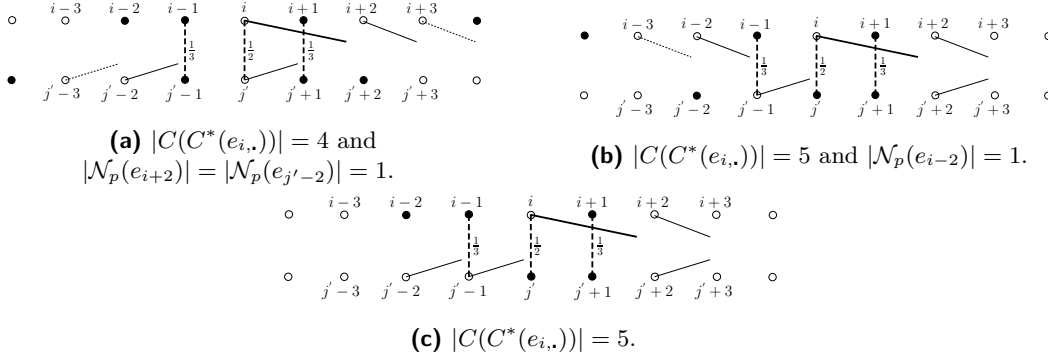


Figure 4.8 The three possible configurations of $\mathcal{N}_p[C(C^*(e_{i,\cdot}))]$ when $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) = (\frac{1}{3}, \frac{1}{2}, \frac{1}{3})$, associated with three possible edge combinations of $C(C^*(e_{i,\cdot}))$ with $|C(C^*(e_{i,\cdot}))| = 4, 5, 5$, respectively. The configuration in Fig. 4.8a also represents the symmetric case where $e_{i-2} \in M$ instead of $e_{j'-2} \in M$ and/or $e_{j'+2} \in M$ instead of $e_{i+2} \in M$.

9. $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$: According to the inequalities (7) and (8) of Lemma 4.6, we have $3 \leq |C(C^*(e_{i,\cdot}))| \leq 4$. If $|C(C^*(e_{i,\cdot}))| = 3$, then the algorithm \mathcal{LS} would replace the three edges of $C(C^*(e_{i,\cdot}))$ by e_{i_1, j_1}^* and the three parallel edges in $C^*(e_{i,\cdot})$ to expand M , a contradiction. Therefore, $|C(C^*(e_{i,\cdot}))| = 4$. From $e_{j'-1}, e_{j'+1} \in M$, we know that either $e_{i+2} \in M$ or $e_{j'+2} \in M$ but not both. If $e_{i+2} \in M$, then all three edges $e_{j'-1}, e_{i,j}, e_{j'+1}$ are singleton edges of M , and the algorithm \mathcal{LS} would replace the four edges of $C(C^*(e_{i,\cdot}))$ by e_{i_1, j_1}^* and the three parallel edges of $C^*(e_{i,\cdot})$ to reduce the singleton edges, a contradiction. Therefore, $e_{j'+2} \in M$, which then implies $|\mathcal{N}_p(e_{j'+2})| = 1$. This only edge combination of $C(C^*(e_{i,\cdot}))$ is shown in Fig. 4.9, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,\cdot}))]$ is also shown.

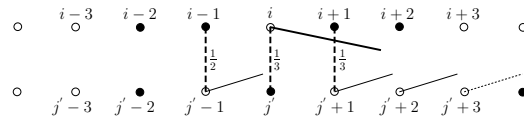
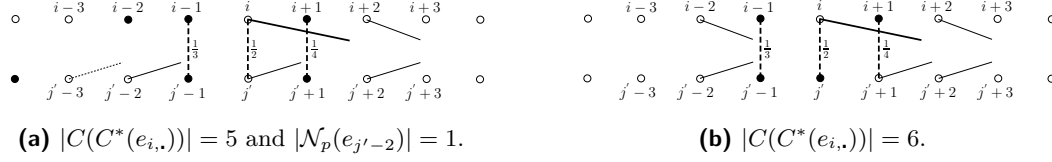


Figure 4.9 The only possible configuration of $\mathcal{N}_p[C(C^*(e_{i,\cdot}))]$ when $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$. We have $|C(C^*(e_{i,\cdot}))| = 4$ and $|\mathcal{N}_p(e_{j'+2})| = 1$.

10. $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) = (\frac{1}{3}, \frac{1}{2}, \frac{1}{4})$: According to the inequalities (7) and (8) of Lemma 4.6, we have $5 \leq |C(C^*(e_{i,\cdot}))| \leq 6$. Note that either $e_{j'} \in M$ or $e_{j'+1} \in M$ but not both. When $e_{j'} \in M$, we have two symmetric cases where $e_{i-2} \in M$ and $e_{j'-2} \in M$, respectively; and we assume $e_{j'-2} \in M$. We conclude that $e_{j'-2}$ must not be a singleton edge of M or $|\mathcal{N}_p(e_{j'-2})| \geq 2$. This edge combination of $C(C^*(e_{i,\cdot}))$ is shown in Fig. 4.10a, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,\cdot}))]$ is also shown. When $e_{j'+1} \in M$, both $e_{i-2} \in M$ and $e_{j'-2} \in M$. This edge combination of $C(C^*(e_{i,\cdot}))$ is shown in Fig. 4.10b, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,\cdot}))]$ is also shown.



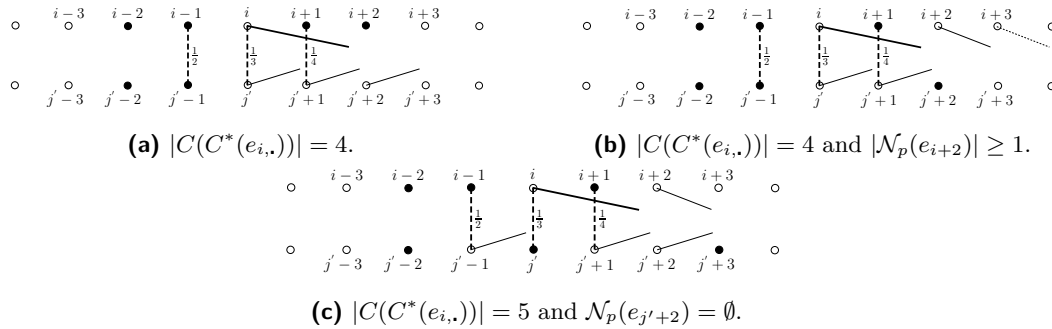
■ **Figure 4.10** The two possible configurations of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ when $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (\frac{1}{3}, \frac{1}{2}, \frac{1}{4})$, associated with two possible edge combinations of $C(C^*(e_{i,.}))$ with $|C(C^*(e_{i,.}))| = 5, 6$, respectively. The configuration in Fig. 4.10a also represents the symmetric case where $e_{i-2} \in M$ instead of $e_{j'-2} \in M$.

11. $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$: According to the inequalities (7) and (8) of Lemma 4.6, we have $4 \leq |C(C^*(e_{i,.}))| \leq 5$. Note that either $e_{j'-1} \notin M$ or $e_{j'} \notin M$ but not both, and $e_{j'+1} \in M$.

When $e_{j'-1} \notin M$, then either $e_{i+2} \in M$ or $e_{j'+2} \in M$ but not both. When $e_{j'+2} \in M$, the edge combination of $C(C^*(e_{i,.}))$ is shown in Fig. 4.11a, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ is also shown.

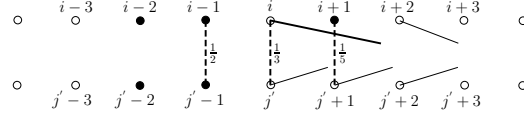
When $e_{i+2} \in M$, we conclude that e_{i+2} should not be a singleton edge of M ; the edge combination of $C(C^*(e_{i,.}))$ is shown in Fig. 4.11b, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ is also shown.

When $e_{j'} \notin M$, then both $e_{i+2} \in M$ and $e_{j'+2} \in M$; we conclude that $\mathcal{N}_p(e_{j'+2}) = \emptyset$. This edge combination of $C(C^*(e_{i,.}))$ is shown in Fig. 4.11c, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ is also shown.



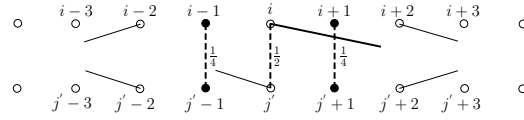
■ **Figure 4.11** The three possible configurations of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ when $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$, associated with three possible edge combinations of $C(C^*(e_{i,.}))$ with $|C(C^*(e_{i,.}))| = 4, 4, 5$, respectively.

12. $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{3})$: This ordered value combination is impossible due to the edge $e_{i,j}$ being a singleton edge of M .
13. $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{5})$: We have $e_{j'}, e_{j'+1}, e_{j'+2}, e_{i+2} \in M$, giving rise to $|C(C^*(e_{i,.}))| = 5$. This only edge combination of $C(C^*(e_{i,.}))$ is shown in Fig. 4.12, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ is also shown.



■ **Figure 4.12** The only possible configuration of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ when $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{5})$, where $|C(C^*(e_{i,.}))| = 5$.

14. $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$: This ordered value combination is impossible due to the edge $e_{i,j}$ being a singleton edge of M .
15. $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$: Since the edge $e_{j'}$ has to be in M , we have both $e_{i-2} \in M$ and $e_{j'-2} \in M$, and both $e_{i+2} \in M$ and $e_{j'+2} \in M$, giving rise to $|C(C^*(e_{i,.}))| = 6$. This only edge combination of $C(C^*(e_{i,.}))$ is shown in Fig. 4.13, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ is also shown.

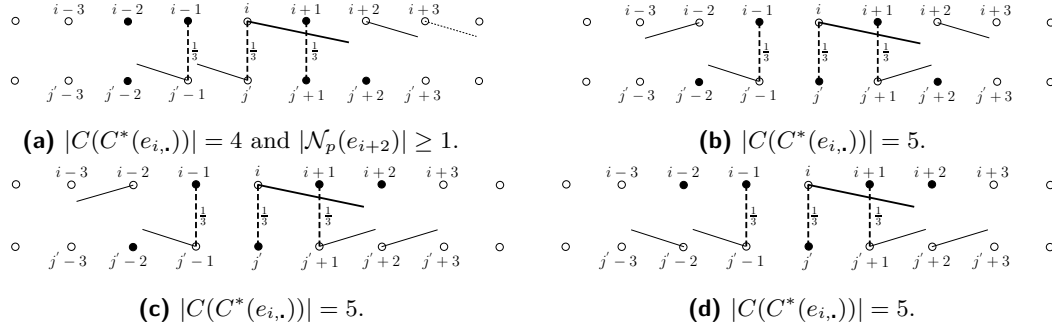


■ **Figure 4.13** The only possible configuration of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ when $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$, where $|C(C^*(e_{i,.}))| = 6$.

16. $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$: According to the inequalities (7) and (8) of Lemma 4.6, we have $4 \leq |C(C^*(e_{i,.}))| \leq 5$. Note that exactly one of the three vertices $j'-1, j, j'+1$ is not incident with any edge of M , we thus consider two cases where $e_{j'} \notin M$ and $e_{j'+1} \notin M$ ($e_{j'-1} \notin M$ is symmetric to $e_{j'+1} \in M$), respectively.

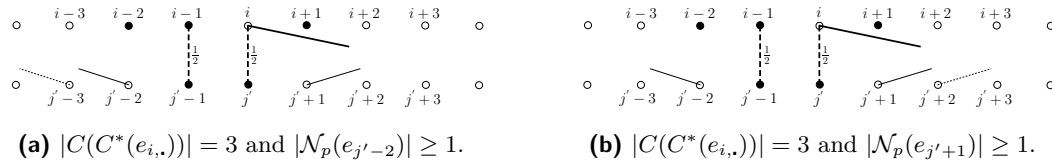
When $e_{j'+1} \notin M$, then either $e_{i+2} \in M$ or $e_{j'+2} \in M$ but not both, while $e_{i-2} \notin M$ and $e_{j'-2} \notin M$. We assume $e_{i+2} \in M$, which implies e_{i+2} should not be a singleton edge of M . This edge combination of $C(C^*(e_{i,.}))$ is shown in Fig. 4.14a, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ is also shown.

When $e_{j'} \notin M$, then either $e_{i+2} \in M$ or $e_{j'+2} \in M$ but not both, and either $e_{i-2} \in M$ or $e_{j'-2} \in M$ but not both. Four different combinations of their memberships of M give rise to 0, 1, 2 singleton edges between $e_{j'-1}$ and $e_{j'+1}$. These three edge combinations of $C(C^*(e_{i,.}))$ are shown in Fig. 4.14b, Fig. 4.14c, Fig. 4.14d, respectively, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ is also shown.



■ **Figure 4.14** The four possible configurations of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ when $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, associated with four possible edge combinations of $C(C^*(e_{i,.}))$ with $|C(C^*(e_{i,.}))| = 4, 5, 5, 5$, respectively. Fig. 4.14a also represents the case where $e_{j'+2} \in M$ instead of $e_{i+2} \in M$; Fig. 4.14c also represents the case where $e_{j'-2}, e_{i+2} \in M$ instead of $e_{i-2}, e_{j'+2} \in M$.

17. $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (\frac{1}{2}, \frac{1}{2}, 0)$: We have $2 \leq |C(C^*(e_{i,.}))| \leq 3$. If $|C(C^*(e_{i,.}))| = 2$, then the algorithm \mathcal{LS} would replace the two edges of $C(C^*(e_{i,.}))$ by e_{i_1, j_1}^* and the two edges of $C^*(e_{i,.})$ to expand M , a contradiction. Therefore, $|C(C^*(e_{i,.}))| = 3$, and furthermore $e_{j'+1} \in M$, and either $e_{i-2} \in M$ or $e_{j'-2} \in M$ but not both. Due to symmetry we assume $e_{j'-2} \in M$. We conclude that at most one of $e_{j'+1}$ and $e_{j'-2}$ can be a singleton edge of M . The edge combination of $C(C^*(e_{i,.}))$ when $e_{j'-2}$ is not a singleton edge is shown in Fig. 4.15a, and the edge combination of $C(C^*(e_{i,.}))$ when $e_{j'+1}$ is not a singleton edge is shown in Fig. 4.15b, where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ is also shown, respectively.



■ **Figure 4.15** The two possible configurations of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ when $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (\frac{1}{2}, \frac{1}{2}, 0)$, associated with the only possible edge combinations of $C(C^*(e_{i,.}))$ with $|C(C^*(e_{i,.}))| = 3$. Each of them also represents the case where $e_{i-2} \in M$ instead of $e_{j'-2} \in M$.

18. $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (\frac{1}{2}, 0, \frac{1}{2})$: We denote the two edges of $C^*(e_{i,.})$ as $e_{i-1, j''-1}$ and $e_{i+1, j''' + 1}$, respectively; clearly, $|(j'' - 1) - (j''' + 1)| \geq 2$. We have $2 \leq |C(C^*(e_{i,.}))| \leq 3$. The same as in the last case, we have $|C(C^*(e_{i,.}))| = 3$, and furthermore exactly one of $i-2, j''-2, j''-1, j''$ is incident with an edge of M , and exactly one of $i+2, j''', j''' + 1, j''' + 2$ is incident with an edge of M . Among these 16 edge combinations of $C(C^*(e_{i,.}))$, in one of them the two edges of $C(C^*(e_{i,.}))$ could be parallel to each other, as shown in Fig. 4.16a (this happens when $j''' = j'' + 1$, and $e_{j''}, e_{j'''} \in M$), where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ is also shown; one of the other 15 is shown in Fig. 4.16b ($j''' > j'' + 1$, $e_{j''-1}, e_{j'''} \in M$), where the corresponding configuration of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ is also shown.

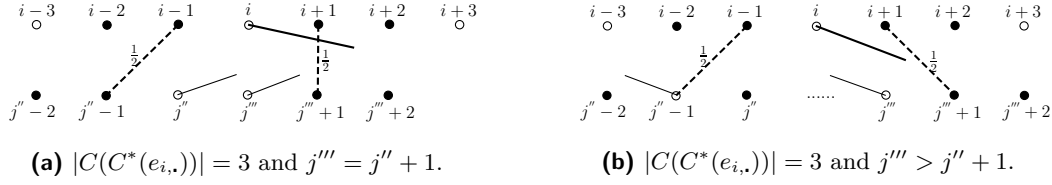


Figure 4.16 The two possible configurations of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ when $\tau(e_{i,j} \leftarrow C^*(e_{i,.})) = (\frac{1}{2}, 0, \frac{1}{2})$, associated with the two possible edge combinations of $C(C^*(e_{i,.}))$ with $|C(C^*(e_{i,.}))| = 3$. The second configuration represents the other 15 symmetric cases exactly one of $i-2, j''-2, j''-1, j''$ is incident with an edge of M and exactly one of $i+2, j''', j'''+1, j'''+2$ is incident with an edge of M , but the two edges are not parallel to each other.

Therefore, we have a total of 27 configurations of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ associated with all the possible edge combinations of $C(C^*(e_{i,.}))$, up to symmetry, for further discussion.

► **Lemma 4.11.** *When $e_{i,j}$ is a singleton edge of M with $\omega(e_{i,j}) \geq 3$, there is at least one parallel edge of M in $C(C^*(e_{i,j}))$ for each of the 8 possible value combinations of $\tau(e_{i,j} \leftarrow C^*(e_{i,j}))$.*

Proof. From Lemma 4.4, for each of the 8 possible value combinations of $\tau(e_{i,j} \leftarrow C^*(e_{i,j}))$, there is an entry 1 in the ordered value combination of $\tau(e_{i,j} \leftarrow C^*(e_{i,.}))$ or $\tau(e_{i,j} \leftarrow C^*(e_{.,j}))$. There are only 5 possible such ordered value combinations, which are $(\frac{1}{2}, 1, \frac{1}{2})$, $(\frac{1}{2}, 1, \frac{1}{3})$, $(1, \frac{1}{2}, \frac{1}{3})$, $(1, \frac{1}{2}, \frac{1}{4})$, and $(\frac{1}{3}, 1, \frac{1}{3})$. The above Figs. 4.1–4.4 show that for the first 4 ordered value combinations, there is at least one parallel edge of M in $C(C^*(e_{i,j}))$.

If $(\frac{1}{3}, 1, \frac{1}{3})$ is the ordered value combination for $\tau(e_{i,j} \leftarrow C^*(e_{i,.}))$, then $\tau(e_{i,j} \leftarrow C^*(e_{.,j}))$ has an ordered value combination either $(\frac{1}{2}, \frac{1}{2}, \frac{1}{3})$ or $(\frac{1}{2}, \frac{1}{3}, \frac{1}{2})$. The above Figs. 4.6 and 4.9 show that there is at least one parallel edge of M in $C(C^*(e_{.,j}))$. This proves the lemma. ◀

4.5 An upper bound on $\omega(e)$ for $e \in C(C^*(e_{i,j})) - \{e_{i,j}\}$

From Lemma 4.10, in the sequel we always consider the case $e_{i,j}$ is a singleton edge of M with $\omega(e_{i,j}) \geq 3$.

We walk through all the 27 configurations of $\mathcal{N}_p[C(C^*(e_{i,.}))]$ to determine an upper bound on $\omega(e)$, for any $e \in C(C^*(e_{i,.})) - \{e_{i,j}\}$.

► **Lemma 4.12.** *For any edge $e \in C(C^*(e_{i,.})) - \{e_{i,j}\}$, $|C(e_{h,\ell}^*)| \geq 2$ for all edges $e_{h,\ell}^* \in C^*(e)$, if any one of the following five conditions holds:*

1. $|C(C^*(e_{i,.}))| = |C^*(e_{i,.})| = 3$.
 2. $|C(C^*(e_{i,.}))| = |C^*(e_{i,.})| + 1 = 4$ and there is an edge $e_{i_1,j_1}^* \in C^*(e_{.,j})$ such that $|C(e_{i_1,j_1}^*)| = 1$.
 3. $e \in C(e_{i_2,j_2}^*)$ for some $e_{i_2,j_2}^* \in C^*(e_{i,.})$ with $|C(e_{i_2,j_2}^*)| = 2$, and there is an edge $e_{i_1,j_1}^* \in C^*(e_{.,j})$ such that $|C(e_{i_1,j_1}^*)| = 1$.
 4. $e \in C(e_{i_1,j_1}^*) \cup C(e_{i_2,j_2}^*)$ for some $e_{i_1,j_1}^*, e_{i_2,j_2}^* \in C^*(e_{i,.})$ with $|C(e_{i_1,j_1}^*) \cup C(e_{i_2,j_2}^*)| = 2$.
 5. $e \in C(e_{i_2,j_2}^*) \cup C(e_{i_3,j_3}^*)$ for some $e_{i_2,j_2}^*, e_{i_3,j_3}^* \in C^*(e_{i,.})$ with $|C(e_{i_2,j_2}^*) \cup C(e_{i_3,j_3}^*)| = 3$, and there is an edge $e_{i_1,j_1}^* \in C^*(e_{.,j})$ such that $|C(e_{i_1,j_1}^*)| = 1$.
- And consequently, $\omega(e) \leq \frac{17}{6}$.

Proof. We prove by contradiction, and thus assume that there is an edge $e_{h,\ell}^* \in C^*(e)$ such that $|C(e_{h,\ell}^*)| = 1$.

If the first condition holds, then the algorithm \mathcal{LS} would replace the three edges of $C(C^*(e_{i,.}))$ by $e_{h,\ell}^*$ and the three parallel edges of $C^*(e_{i,.})$ to expand M , a contradiction.

If the second condition holds, then the algorithm \mathcal{LS} would replace the four edges of $C(C^*(e_{i,\cdot}))$ by e_{i_1,j_1}^* , $e_{h,\ell}^*$, and the three parallel edges in $C^*(e_{i,\cdot})$ to expand M , a contradiction.

If the third condition holds, then the algorithm \mathcal{LS} would replace the two edges of $C(e_{i_2,j_2}^*)$ by e_{i_2,j_2}^* , e_{i_1,j_1}^* , $e_{h,\ell}^*$ to expand M , a contradiction.

If the fourth condition holds, then the algorithm \mathcal{LS} would replace the two edges of $C(e_{i_1,j_1}^*) \cup C(e_{i_2,j_2}^*)$ by e_{i_1,j_1}^* , e_{i_2,j_2}^* , $e_{h,\ell}^*$ to expand M , a contradiction.

If the fifth condition holds, then the algorithm \mathcal{LS} would replace the three edges of $C(e_{i_2,j_2}^*) \cup C(e_{i_3,j_3}^*)$ by e_{i_2,j_2}^* , e_{i_3,j_3}^* , e_{i_1,j_1}^* , $e_{h,\ell}^*$ to expand M , a contradiction.

Therefore, we proved that $|C(e_{h,\ell}^*)| \geq 2$ for all edges $e_{h,\ell}^* \in C^*(e)$. It then follows from Lemma 4.3 that $\omega(e) \leq 5 \times \frac{1}{2} + \frac{1}{3} = \frac{17}{6}$. \blacktriangleleft

► **Lemma 4.13.** *For each edge $e \in C(C^*(e_{i,\cdot})) - \{e_{i,j}\}$ in Figs. 4.1, 4.3, 4.6a, 4.6b, 4.8a, 4.9, 4.11a, 4.11b, 4.14a, 4.15a, 4.15b, 4.16a and 4.16b, $e_{j'-2}$ in Fig. 4.7a, $e_{j'-1}$ in Fig. 4.8b, $e_{j'}$ in Fig. 4.10a, $e_{j'-1}$ in Fig. 4.11c, and $e_{j'}$ in Fig. 4.13, its total amount of tokens is $\omega(e) \leq \frac{17}{6}$.*

Proof. At least one of the five conditions in Lemma 4.12 applies to each of these edges. For example, in Fig. 4.1, for the edge $e_{j'-2}$, the fourth condition of Lemma 4.12 holds by setting $(i_1, j_1) := (i-1, j'-1)$ and $(i_2, j_2) := (i, j')$; for the edge e_{i+2} , the fourth condition of Lemma 4.12 holds by setting $(i_1, j_1) := (i, j')$ and $(i_2, j_2) := (i+1, j'+1)$. \blacktriangleleft

► **Lemma 4.14.** *For both the edges $e_{i+q}, e_{j'+q} \in C(C^*(e_{i,\cdot}))$ shown in Figs. 4.2, 4.4, 4.5, 4.7a, 4.8b, 4.8c, 4.10b and 4.11c, for some $q = 2$ or -2 , the total amount of tokens for each of them is at most $\frac{35}{12}$.*

Proof. Consider the edge e_{i+q} . If it is a parallel edge of M , then it simply cannot fit into any of the 27 configurations shown in Figs. 4.1–4.16, in which the edge $e_{i,j}$ is a singleton edge of M . (By “fitting into” it means the edge e_{i+q} takes up the role of the edge $e_{i,j}$ in the configuration.) If e_{i+q} is a singleton edge of M , we show next that due to the existence of the paired edge $e_{j'+q} \in M$, e_{i+q} cannot fit into any of the 27 configurations shown in Figs. 4.1–4.16 either. This is done by using the edge combinations of $C(C^*(e_{i,\cdot}))$ and the existence of certain edges in $\mathcal{N}_p(C(C^*(e_{i,\cdot})))$.

In more details, we first see that e_{i+q} can only possibly fit into 7 of the 27 configurations shown in Figs. 4.8a, 4.10a, 4.11a, 4.11b, 4.12, 4.13 and 4.14a, due to the existence of the edge $e_{j'+q} \in M$. Next, if it were fit in any of them, then in the fitted configuration there is an edge $e_{i-2} \in M$ but none of the five edges $e_{i-3}, e_{j'-3}, e_{j'-2}, e_{i-1}, e_{j'-1}$ can be in M . This last requirement rules out Fig. 4.8a due to $e_{j'-3}, e_{i+3} \in \mathcal{N}_p(C(C^*(e_{i,\cdot})))$; it rules out Fig. 4.10a due to $e_{j'-3} \in \mathcal{N}_p(C(C^*(e_{i,\cdot})))$; it rules out Figs. 4.11a, 4.11b and 4.12 due to $e_{j'+1} \in C(C^*(e_{i,\cdot}))$ but none of $e_{i-2}, e_{j'-2}$ is in $C(C^*(e_{i,\cdot}))$; it rules out Fig. 4.13 due to $e_{i-2}, e_{j'-2}, e_{i+2}, e_{j'+2} \in C(C^*(e_{i,\cdot}))$; and it rules out Fig. 4.14a due to $e_{j'-1} \in C(C^*(e_{i,\cdot}))$ and $e_{i+3} \in \mathcal{N}_p(C(C^*(e_{i,\cdot})))$.

Therefore, $\omega(e_{i+q}) < 3$.

Using at most six values from $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\}$, the sum closest but less than 3 is $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} = \frac{59}{20}$. In order for $\tau(e_{i,j} \leftarrow C^*(e_{i,j}))$ to have a value combination $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}\}$, Lemma 4.2 says that the value combinations for $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot}))$ and $\tau(e_{i,j} \leftarrow C^*(e_{\cdot,j}))$ are $\{1, \frac{1}{2}, \frac{1}{2}\}$ and $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{5}\}$. Furthermore, Lemmas 4.2 and 4.6 together state that the subsequent ordered value combinations are $(\frac{1}{2}, 1, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{4}, \frac{1}{5})$. However, $(\frac{1}{2}, 1, \frac{1}{2})$ requires $e_{i,j}$ to be a singleton edge of M , while $(\frac{1}{2}, \frac{1}{4}, \frac{1}{5})$ implies $e_{i,j}$ is a parallel edge of M , a contradiction.

The second closest sum to 3 is $\frac{35}{12}$, that is the sum of the value combinations $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}\}$ (which can be ruled out similarly as in the last paragraph) and $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}\}$. Therefore, $\omega(e_{i+q}) \leq \frac{35}{12}$. \blacktriangleleft

► **Lemma 4.15.** *For each edge $e \in C(C^*(e_{i,j})) - \{e_{i,j}\}$ with $\omega(e_{i,j}) \geq 3$, we have $\omega(e) \leq \frac{35}{12}$, except for the following two cases where we have $\omega(e_{i,j}) = 3$:*

1. *in the configuration shown in Fig. 4.14b, it is possible to have either $\omega(e_{j'-1}) = 3$ (when $|\mathcal{N}_p(e_{i-2})| \geq 1$) or $\omega(e_{j'+1}) = 3$ (when $|\mathcal{N}_p(e_{i+2})| \geq 1$), but not both;*
2. *in the configuration shown in Fig. 4.14c, it is possible to have either $\omega(e_{j'-1}) = 3$ or $\omega(e_{i-2}) = 3$ (when $|\mathcal{N}_p(e_{j'-3})| \geq 1$), but not both.*

Proof. Recall that Lemma 4.13 settled all the edges of $C(C^*(e_{i,j})) - \{e_{i,j}\}$ in Figs. 4.1, 4.3, 4.6a, 4.6b, 4.8a, 4.9, 4.11a, 4.11b, 4.14a, 4.15a, 4.15b, 4.16a and 4.16b, $e_{j'-2}$ in Fig. 4.7a, $e_{j'-1}$ in Fig. 4.8b, $e_{j'}$ in Fig. 4.10a, $e_{j'-1}$ in Fig. 4.11c, and $e_{j'}$ in Fig. 4.13; Lemma 4.14 settled all the paired edges $e_{i+q}, e_{j'+q} \in C(C^*(e_{i,j}))$ in Figs. 4.2, 4.4, 4.5, 4.7a, 4.8b, 4.8c, 4.10b and 4.11c, for some $q = 2$ or -2 , and all the edges known to be parallel, including $e_{j'-2}$ in Fig. 4.2, $e_{j'+1}$ in Fig. 4.4, $e_{j'+1}$ in Fig. 4.7a, e_{i-2} in Fig. 4.8b, $e_{j'-1}$ in Fig. 4.8c, $e_{j'-2}$ in Fig. 4.10a, $e_{j'+1}$ in Fig. 4.10b, $e_{j'+1}$ in Fig. 4.11c, $e_{j'}, e_{j'+1}, e_{j'+2}$ in Fig. 4.12, $e_{j'+1}, e_{j'+2}$ in Fig. 4.14c, and $e_{j'-2}, e_{j'-1}, e_{j'+1}, e_{j'+2}$ in Fig. 4.14d.

We therefore are left to prove the lemma for the edges not known to be parallel in Figs. 4.10a, 4.12, 4.13, 4.14b and 4.14c. We deal with them separately in the following.

1. The edges $e_{i+2}, e_{j'+2}$ in Fig. 4.10a and the edges $e_{i-2}, e_{j'-2}, e_{i+2}, e_{j'+2}$ in Fig. 4.13, which can be settled the same.

Consider the edge e_{i+2} , which can potentially fit into the configuration in Fig. 4.10a or Fig. 4.13. In either case, there is an edge $e_{i_1, j_1}^* \in C^*(e_{i,j})$ such that $|C(e_{i_1, j_1}^*)| = 1$ and there is an edge $e_{h_1, \ell_1}^* \in C^*(e_{i+2})$ such that $|C(e_{h_1, \ell_1}^*)| = 1$. Then the algorithm \mathcal{LS} would replace the four edges $e_{i,j}, e_{j'}, e_{i+2}, e_{j'+2}$ by the five edges $e_{i,j'}, e_{i+1, j'+1}^*, e_{i+2, j'+2}^*, e_{i_1, j_1}^*, e_{h_1, \ell_1}^*$ to expand M , a contradiction. In summary, e_{i+2} cannot fit into any of the 27 configurations shown in Figs. 4.1–4.16 and thus $\omega(e_{i+2}) \leq \frac{35}{12}$.

2. The edge e_{i+2} in Fig. 4.12.

If e_{i+2} is to fit in, then it can fit only into the configuration in Fig. 4.12. This suggests that $C(C^*(e_{i+2})) = C(C^*(e_{i,j}))$. Since there is an edge $e_{i_1, j_1}^* \in C^*(e_{i,j})$ such that $|C(e_{i_1, j_1}^*)| = 1$ and there is an edge $e_{h_1, \ell_1}^* \in C^*(e_{i+2})$ such that $|C(e_{h_1, \ell_1}^*)| = 1$, the algorithm \mathcal{LS} would replace the five edges of $C(C^*(e_{i,j}))$ by any six edges from $\{e_{i-1, j'-1}^*, e_{i,j'}, e_{i+1, j'+1}^*, e_{i+2, j'+2}^*, e_{i+3, j'+3}^*, e_{i_1, j_1}^*, e_{h_1, \ell_1}^*\}$ to expand M , a contradiction. In summary, e_{i+2} cannot fit into any of the 27 configurations shown in Figs. 4.1–4.16 and thus $\omega(e_{i+2}) \leq \frac{35}{12}$.

3. The edges e_{i-2} and e_{i+2} in Fig. 4.14b, which can be settled the same.

Consider the edge e_{i-2} , which can potentially fit into the configuration in Fig. 4.14b or Fig. 4.14c. In either case, all the four edges $e_{i-2}, e_{j'-1}, e_{i,j}, e_{j'+1}$ are singleton edges of M , and there is an edge $e_{i_1, j_1}^* \in C^*(e_{i,j})$ such that $|C(e_{i_1, j_1}^*)| = 1$ and there is an edge $e_{h_1, \ell_1}^* \in C^*(e_{i-2})$ such that $|C(e_{h_1, \ell_1}^*)| = 1$. Then the algorithm \mathcal{LS} would replace these four singleton edges of M by the edges $e_{i_1, j_1}^*, e_{h_1, \ell_1}^*$ and the two parallel edges $e_{i-1, j'-1}^*, e_{i,j'}$ to reduce the singleton edges of M , a contradiction. In summary, e_{i-2} cannot fit into any of the 27 configurations shown in Figs. 4.1–4.16 and thus $\omega(e_{i-2}) \leq \frac{35}{12}$.

4. The edges $e_{j'-1}$ and $e_{j'+1}$ in Fig. 4.14b, which can be settled the same.

Consider the edge $e_{j'-1}$, which can potentially fit into the configuration in Fig. 4.14b or Fig. 4.14c. If $e_{j'-1}$ fits into the configuration in Fig. 4.14b, then the same as in the last case the algorithm \mathcal{LS} would be able to reduce the singleton edges of M , a contradiction.

If $e_{j'-1}$ fits into the configuration in Fig. 4.14c, then the edge e_{i-2} is a parallel edge of M . From $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, we conclude that $\omega(e_{i,j}) \leq 3$, and consequently $\omega(e_{i,j}) = 3$ and $\omega(e_{j'-1}) = 3$.

It is easy to see that we cannot have both $\omega(e_{j'-1}) = \omega(e_{j'+1}) = 3$, since otherwise the algorithm \mathcal{LS} would be able to expand M by swapping out the five edges of $C(C^*(e_{i,\cdot}))$, a contradiction.

In summary, we have either $\omega(e_{j'-1}) \leq \frac{35}{12}$ or $\omega(e_{j'-1}) = 3$, the latter of which implies $|\mathcal{N}_p(e_{i-2})| \geq 1$ and it is the first case stated in the lemma.

5. The edge e_{i-2} in Fig. 4.14c.

If e_{i-2} is to fit in, then it can fit only into the configuration in Fig. 4.14b or Fig. 4.14c. If e_{i-2} fits into the configuration in Fig. 4.14b, then all the four edges $e_{i,j}$, $e_{j'-1}$, e_{i-2} , $e_{j'-3}$ are singleton edges of M . Since there is an edge $e_{i_1,j_1}^* \in C^*(e_{i,j})$ such that $|C(e_{i_1,j_1}^*)| = 1$ and there is an edge $e_{h_1,\ell_1}^* \in C^*(e_{i-2})$ such that $|C(e_{h_1,\ell_1}^*)| = 1$, the algorithm \mathcal{LS} would replace these four singleton edges of M by the edges e_{i_1,j_1}^* , e_{h_1,ℓ_1}^* and the two parallel edges $e_{i-1,j'-1}^*$, $e_{i-2,j'-2}^*$ to reduce the singleton edges, a contradiction. If e_{i-2} fits into the configuration in Fig. 4.14c, then the edge $e_{j'-3}$ is a parallel edge of M . From $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, we conclude that $\omega(e_{i,j}) \leq 3$, and consequently $\omega(e_{i,j}) = 3$ and $\omega(e_{i-2}) = 3$. In summary, we have either $\omega(e_{i-2}) \leq \frac{35}{12}$ or $\omega(e_{i-2}) = 3$, the latter of which implies $|\mathcal{N}_p(e_{j'-3})| \geq 1$.

6. The edge $e_{j'-1}$ in Fig. 4.14c.

If $e_{j'-1}$ is to fit in, then it can fit only into the configuration in Fig. 4.14b or Fig. 4.14c. If $e_{j'-1}$ fits into the configuration in Fig. 4.14b, then the edge e_{i-2} is a singleton edge of M . If $e_{j'-1}$ fits into the configuration in Fig. 4.14c, then the edge e_{i-2} is a parallel edge of M . From $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot})) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, we conclude that $\omega(e_{i,j}) \leq 3$, and consequently $\omega(e_{i,j}) = 3$ and $\omega(e_{j'-1}) = 3$. In summary, we have either $\omega(e_{j'-1}) \leq \frac{35}{12}$ or $\omega(e_{j'-1}) = 3$.

It is also easy to see that we cannot have both $\omega(e_{j'-1}) = \omega(e_{i-2}) = 3$ in the last two items, since otherwise the algorithm \mathcal{LS} would be able to expand M by swapping out the five edges of $C(C^*(e_{i,\cdot}))$, a contradiction. This is the second case stated in the lemma. We have proved the lemma. \blacktriangleleft

4.6 An upper bound on $\omega(e)$ for $e \in C(C^*(e_{i,j}))$ known to be parallel

In this section, we provide a better upper bound on the total amount of tokens received by an edge of $C(C^*(e_{i,j}))$ that is *known* to be parallel, for example, in Fig. 4.2 the edge $e_{j'-2}$ is known parallel but the edge e_{i+2} is not. Also, from Lemma 4.15, in Fig. 4.14b it is possible to have $\omega(e_{j'-1}) = 3$ when $|\mathcal{N}_p(e_{i-2})| \geq 1$; we therefore consider the edge e_{i-2} to be parallel too. For the same reason, we consider the edge e_{i+2} in Fig. 4.14b to be parallel.

► **Lemma 4.16.** *For each parallel edge $e \in C(C^*(e_{i,\cdot}))$ with $\omega(e_{i,j}) \geq 3$, we have $|C(e_{h,\ell}^*)| \geq 2$ for all $e_{h,\ell}^* \in C^*(e)$, except for the following edges:*

1. the edge $e_{j'+2}$ in Figs. 4.4, 4.7a, 4.10b, 4.11c and 4.12,
2. the edges e_{i-2}, e_{i+2} in Fig. 4.14b,
3. the edges $e_{j'+1}, e_{j'+2}$ in Fig. 4.14c, and
4. the edges $e_{j'-1}, e_{j'-2}, e_{j'+1}, e_{j'+2}$ in Fig. 4.14d.

Proof. At least one of the five conditions in Lemma 4.12 applies to each of these edges. For example, in Fig. 4.2, for the edge $e_{j'-2}$, the fourth condition of Lemma 4.12 holds by setting $(i_1, j_1) := (i-1, j'-1)$ and $(i_2, j_2) := (i, j')$. \blacktriangleleft

Among all the 27 configurations in Figs. 4.1–4.16, we have the following two observations.

► **Observation 4.1.** If the edge e_{i-1} (or the edge $e_{j'-1}$) is a known parallel edge of M in $C(C^*(e_{i,\cdot}))$, and $e_{i-1,j'-1}^* \in M^*$, then $|C(e_{i-1,j'-1}^*)| \geq 3$; if the edge $e_{j'}$ is a known parallel edge of M in $C(C^*(e_{i,\cdot}))$, and $e_{i,j'}^* \in M^*$, then $|C(e_{i,j'}^*)| \geq 3$; if the edge e_{i+1} (or the edge $e_{j'+1}$) is a known parallel edge of M in $C(C^*(e_{i,\cdot}))$, and $e_{i+1,j'+1}^* \in M^*$, then $|C(e_{i+1,j'+1}^*)| \geq 3$.

► **Observation 4.2.** When $e_{i,j'}^* \in C^*(e_{i,\cdot})$, if the edge e_{i+p} (or $e_{j'+p}$, respectively) is a known parallel edge of M in $C(C^*(e_{i,\cdot}))$ for some $p = -2, 2$, then $|C(e_{i+p}^*)| \geq 2$ (or $|C(e_{j'+p}^*)| \geq 2$, respectively).

Based on Lemmas 4.10, 4.12, and Observations 4.1 and 4.2, we can prove the following two lemmas.

► **Lemma 4.17.** For any pair of parallel edges $e_{h,\ell}, e_{h+1,\ell+1} \in C(C^*(e_{i,\cdot}))$, and an edge $e^* \in C^*(e_{h,\ell}) \cap C^*(e_{h+1,\ell+1})$, we have $|C(e^*)| \geq 3$ if one of the following three conditions holds.

1. $|C(C^*(e_{i,\cdot}))| = |C^*(e_{i,\cdot})| = 3$.
2. $|C(C^*(e_{i,\cdot}))| = |C^*(e_{i,\cdot})| + 1 = 4$ and there is an edge $e_{i_1,j_1}^* \in C^*(e_{i,\cdot})$ such that $|C(e_{i_1,j_1}^*)| = 1$.
3. $e_{h,\ell}, e_{h+1,\ell+1} \in C(e_{i_2,j_2}^*) \cup C(e_{i_3,j_3}^*)$ for some $e_{i_2,j_2}^*, e_{i_3,j_3}^* \in C^*(e_{i,\cdot})$ with $|C(e_{i_2,j_2}^*) \cup C(e_{i_3,j_3}^*)| = 3$, and there is an edge $e_{i_1,j_1}^* \in C^*(e_{i,\cdot})$ such that $|C(e_{i_1,j_1}^*)| = 1$.

Proof. We prove by contradiction, and thus assume that there is an edge $e^* \in C^*(e_{h,\ell}) \cap C^*(e_{h+1,\ell+1})$ such that $|C(e^*)| = 2$.

If the first condition holds, then it follows from Observation 4.1 that $e^* \notin C^*(e_{i,\cdot})$. In this case, the algorithm \mathcal{LS} would replace the three edges of $C(C^*(e_{i,\cdot}))$ by e^* and the three edges of $C^*(e_{i,\cdot})$ to expand M , a contradiction.

If the second condition holds, then again it follows from Observation 4.1 that $e^* \notin C^*(e_{i,\cdot})$. Also, the edge e_{i_1,j_1}^* is distinct from e^* . In this case, the algorithm \mathcal{LS} would replace the four edges in $C(C^*(e_{i,\cdot}))$ by e_{i_1,j_1}^*, e^* , and the three edges of $C^*(e_{i,\cdot})$ to expand M , a contradiction.

If the third condition holds, then by Observation 4.1 the edge e^* is distinct from $e_{i_2,j_2}^*, e_{i_3,j_3}^*$, and the edge e_{i_1,j_1}^* is distinct from e^* . In this case, the algorithm \mathcal{LS} would replace the three edges of $C(e_{i_2,j_2}^*) \cup C(e_{i_3,j_3}^*)$ by $e_{i_2,j_2}^*, e_{i_3,j_3}^*, e_{i_1,j_1}^*, e^*$ to expand M , a contradiction. ◀

► **Lemma 4.18.** For any pair of parallel edges $e_{h,\ell}, e_{h+1,\ell+1}$ where $e_{h,\ell} \in C(C^*(e_{i,\cdot}))$, there is at most one edge $e^* \in C^*(e_{h,\ell}) \cap C^*(e_{h+1,\ell+1})$ such that $|C(e^*)| = 2$, if one of the following six conditions holds.

1. $e_{h+1,\ell+1} \notin C(C^*(e_{i,\cdot}))$ and $|C(C^*(e_{i,\cdot}))| = |C^*(e_{i,\cdot})| = 3$.
2. $e_{h,\ell} \in C(e_{i_1,j_1}^*) \cup C(e_{i_2,j_2}^*)$, $e_{h+1,\ell+1} \notin C(e_{i_1,j_1}^*) \cup C(e_{i_2,j_2}^*)$ for some $e_{i_1,j_1}^*, e_{i_2,j_2}^* \in C^*(e_{i,\cdot})$ with $|C(e_{i_1,j_1}^*) \cup C(e_{i_2,j_2}^*)| = 2$.
3. $e_{h+1,\ell+1} \notin C(C^*(e_{i,\cdot}))$, $|C(C^*(e_{i,\cdot}))| = |C^*(e_{i,\cdot})| + 1 = 4$, and there is an edge $e_{i_1,j_1}^* \in C^*(e_{i,\cdot})$ such that $|C(e_{i_1,j_1}^*)| = 1$.
4. $e_{h,\ell} \in C(e_{i_2,j_2}^*) \cup C(e_{i_3,j_3}^*)$, $e_{h+1,\ell+1} \notin C(e_{i_2,j_2}^*) \cup C(e_{i_3,j_3}^*)$ for some $e_{i_2,j_2}^*, e_{i_3,j_3}^* \in C^*(e_{i,\cdot})$ with $|C(e_{i_2,j_2}^*) \cup C(e_{i_3,j_3}^*)| = 3$, and there is an edge $e_{i_1,j_1}^* \in C^*(e_{i,\cdot})$ such that $|C(e_{i_1,j_1}^*)| = 1$.
5. $e_{h,\ell} \in C(e_{i_2,j_2}^*)$, $e_{h+1,\ell+1} \notin C(e_{i_2,j_2}^*)$ for some $e_{i_2,j_2}^* \in C^*(e_{i,\cdot})$ with $|C(e_{i_2,j_2}^*)| = 2$, and there is an edge $e_{i_1,j_1}^* \in C^*(e_{i,\cdot})$ such that $|C(e_{i_1,j_1}^*)| = 1$.
6. $e_{h+1,\ell+1} \in C(C^*(e_{i,\cdot}))$, $|C(C^*(e_{i,\cdot}))| = |C^*(e_{i,\cdot})| + 2 = 5$, and there is an edge $e_{i_1,j_1}^* \in C^*(e_{i,\cdot})$ such that $|C(e_{i_1,j_1}^*)| = 1$.

Proof. We prove by contradiction, and thus assume that there are two edges $e_{h_1, \ell_1}^*, e_{h_2, \ell_2}^* \in C^*(e_{h, \ell}) \cap C^*(e_{h+1, \ell+1})$ such that $C(e_{h_1, \ell_1}^*) = C(e_{h_2, \ell_2}^*) = \{e_{h, \ell}, e_{h+1, \ell+1}\}$.

If the first condition holds, then due to $e_{h+1, \ell+1} \notin C(C^*(e_{i, \cdot}))$, none of $e_{h_1, \ell_1}^*, e_{h_2, \ell_2}^*$ is in $C^*(e_{i, \cdot})$. In this case, the algorithm \mathcal{LS} would replace the edge $e_{h+1, \ell+1}$ and the three edges of $C(C^*(e_{i, \cdot}))$ by $e_{h_1, \ell_1}^*, e_{h_2, \ell_2}^*$ and the three edges of $C^*(e_{i, \cdot})$ to expand M , a contradiction.

The other five conditions can be similarly proved by this kind of contradiction. \blacktriangleleft

Using Lemmas 4.17 and 4.18, we can prove a better upper bound on $\omega(e)$ for those edges stated in Lemma 4.16. This better bound is $\frac{5}{2}$, a reduction from $\frac{17}{6}$ stated in Lemma 4.12.

► **Lemma 4.19.** *For any parallel edge $e \in C(C^*(e_{i, \cdot}))$ discussed in Lemma 4.16, its total amount of tokens received $\omega(e)$ can be better bounded, in particular, $\omega(e) \leq \frac{5}{2}$.*

Proof. We enumerate through all these edges in the following:

1. In Fig. 4.1, we have $\tau(e_{i, j} \leftarrow C^*(e_{i, \cdot})) = (\frac{1}{2}, 1, \frac{1}{2})$. For the edge $e_{j'-2}$, it is parallel to $e_{j'-3} \notin C(C^*(e_{i, \cdot}))$. By the condition 1 of Lemma 4.18 and Lemma 4.16, $\omega(e_{j'-2}) \leq 3(\frac{1}{2} + \frac{1}{3}) = \frac{5}{2} = 2.5$. The same argument applies to the edge e_{i+2} .
In the rest of the proof, we point out only the condition used in the argument to avoid repetition.
2. In Fig. 4.2, $\omega(e_{j'-2}) \leq 3(\frac{1}{2} + \frac{1}{3}) = \frac{5}{2} = 2.5$, due to the condition 2 of Lemma 4.18.
3. In Fig. 4.3, $\omega(e_{j'+1}), \omega(e_{j'+2}) \leq 2(\frac{1}{2} + 2 \times \frac{1}{3}) = \frac{7}{3} \approx 2.333$, due to the condition 1 of Lemma 4.17.
4. In Fig. 4.4, $\omega(e_{j'+1}) \leq (\frac{1}{2} + \frac{1}{4} + \frac{1}{3}) + (2 \times \frac{1}{2} + \frac{1}{3}) = \frac{29}{12} \approx 2.417$, due to the condition 2 of Lemma 4.17.
5. In Fig. 4.6a, $\omega(e_{j'+1}), \omega(e_{j'+2}) \leq 2(\frac{1}{2} + 2 \times \frac{1}{3}) = \frac{7}{3} \approx 2.333$, due to the condition 2 of Lemma 4.17.
6. In Fig. 4.6b, $\omega(e_{j'-2}) \leq 3(\frac{1}{2} + \frac{1}{3}) = \frac{5}{2} = 2.5$, due to the condition 3 of Lemma 4.18;
 $\omega(e_{i+2}) \leq 4 \times \frac{1}{3} + 2 \times \frac{1}{2} = \frac{7}{3} \approx 2.333$, due to the condition 3 of Lemma 4.18.
7. In Fig. 4.7a, $\omega(e_{j'+1}) \leq (\frac{1}{2} + \frac{1}{4} + \frac{1}{3}) + (2 \times \frac{1}{2} + \frac{1}{3}) = \frac{29}{12} \approx 2.417$, due to the condition 4 of Lemma 4.18.
8. In Fig. 4.8a, $\omega(e_{j'-2}), \omega(e_{i+2}) \leq 4 \times \frac{1}{3} + 2 \times \frac{1}{2} = \frac{7}{3} \approx 2.333$, due to the condition 3 of Lemma 4.18.
9. In Fig. 4.8b, $\omega(e_{j'-1}), \omega(e_{j'-2}) \leq 2(\frac{1}{2} + 2 \times \frac{1}{3}) = \frac{7}{3} \approx 2.333$, due to the condition 3 of Lemma 4.17.
10. In Fig. 4.8c, $\omega(e_{i-2}) \leq 2(\frac{1}{2} + 2 \times \frac{1}{3}) = \frac{7}{3} \approx 2.333$, due to the condition 4 of Lemma 4.18.
11. In Fig. 4.9, $\omega(e_{j'+1}), \omega(e_{j'+2}) \leq 5 \times \frac{1}{3} + \frac{1}{2} = \frac{13}{6} \approx 2.167$, due to the condition 2 of Lemma 4.17.
12. In Fig. 4.10a, $\omega(e_{j'-2}) \leq 4 \times \frac{1}{3} + 2 \times \frac{1}{2} = \frac{7}{3} \approx 2.333$, due to the condition 4 of Lemma 4.18.
13. In Fig. 4.10b, $\omega(e_{j'+1}) \leq (\frac{1}{2} + \frac{1}{4} + \frac{1}{3}) + (2 \times \frac{1}{2} + \frac{1}{3}) = \frac{29}{12} \approx 2.417$, due to the condition 5 of Lemma 4.18.
14. In Fig. 4.11a, $\omega(e_{j'}), \omega(e_{j'+2}) \leq (\frac{1}{2} + \frac{1}{3} + \frac{1}{4}) + (\frac{1}{2} + 2 \times \frac{1}{3}) = \frac{9}{4} = 2.25$, due to the condition 2 of Lemma 4.17;
 $\omega(e_{j'+1}) \leq (\frac{1}{3} + \frac{1}{4} + \frac{1}{3}) + (\frac{1}{2} + 2 \times \frac{1}{3}) = \frac{25}{12} \approx 2.083$, due to the condition 2 of Lemma 4.17.
15. In Fig. 4.11b, $\omega(e_{j'}), \omega(e_{i+2}) \leq (\frac{1}{2} + \frac{1}{3} + \frac{1}{4}) + (\frac{1}{2} + 2 \times \frac{1}{3}) = \frac{9}{4} = 2.25$, due to the condition 2 of Lemma 4.17;
 $\omega(e_{j'+1}) \leq (\frac{1}{3} + \frac{1}{4} + \frac{1}{3}) + (\frac{1}{2} + 2 \times \frac{1}{3}) = \frac{25}{12} \approx 2.083$, due to the condition 2 of Lemma 4.17.

16. In Fig. 4.11c, $\omega(e_{j'+1}) \leq (\frac{1}{2} + \frac{1}{3} + \frac{1}{4}) + (\frac{1}{2} + 2 \times \frac{1}{3}) = \frac{9}{4} = 2.25$, due to the condition 4 of Lemma 4.18.
17. In Fig. 4.12, $\omega(e_{j'}) \leq (\frac{1}{2} + \frac{1}{3} + \frac{1}{5}) + (\frac{1}{2} + 2 \times \frac{1}{3}) = \frac{11}{5} = 2.2$, due to the condition 3 of Lemma 4.17;
 $\omega(e_{j'+1}) \leq (2 \times \frac{1}{3} + \frac{1}{5}) + (\frac{1}{2} + 2 \times \frac{1}{3}) = \frac{61}{30} \approx 2.033$, due to the condition 3 of Lemma 4.17.
18. In Fig. 4.14a, $\omega(e_{j'}) \leq 5 \times \frac{1}{3} + \frac{1}{2} = \frac{13}{6} \approx 2.167$, due to the condition 2 of Lemma 4.17;
 $\omega(e_{j'-1}) \leq 2(\frac{1}{2} + 2 \times \frac{1}{3}) = \frac{7}{3} \approx 2.333$, due to the condition 2 of Lemma 4.17;
 $\omega(e_{i+2}) \leq 4 \times \frac{1}{3} + 2 \times \frac{1}{2} = \frac{7}{3} \approx 2.333$, due to the condition 3 of Lemma 4.18.
19. In Fig. 4.15a, $\omega(e_{j'-2}) \leq 3 \times \frac{1}{2} + 3 \times \frac{1}{3} = \frac{5}{2} = 2.5$, due to the condition 4 of Lemma 4.18.
20. In Fig. 4.15b, $\omega(e_{j'+1}) \leq 2 \times \frac{1}{2} + 3 \times \frac{1}{3} = 2$, due to the condition 4 of Lemma 4.18 and no edge of M^* incident at $j' + 1$.
21. In Fig. 4.16a, $\omega(e_{j''}), \omega(e_{j'''}) \leq 1 + 3 \times \frac{1}{2} = \frac{5}{2} = 2.5$, simply due to no edge of M^* incident at j'' and $e_{j'''}$ where $j''' = j'' + 1$.

Note the maximum value among the above is $\frac{5}{2} = 2.5$. The lemma is proved. \blacktriangleleft

The next lemma is on the parallel edges excluded from Lemma 4.19.

► **Lemma 4.20.** *For each of following parallel edge $e \in C(C^*(e_{i,\cdot}))$ with $\omega(e_{i,j}) \geq 3$, we have*

1. *for the edge $e_{j'+2}$ in Figs. 4.4, 4.7a and 4.10b, $\omega(e_{j'+2}) \leq \frac{29}{12}$;*
2. *for the edge $e_{j'+2}$ in Figs. 4.11c and 4.12, $\omega(e_{j'+2}) \leq \frac{35}{12}$;*
3. *for the edges e_{i-2}, e_{i+2} in Fig. 4.14b, either $\omega(e_{i-2}), \omega(e_{i+2}) \leq \frac{35}{12}$, or $\omega(e_{i-2}) \leq \frac{13}{6}$ when $\omega(e_{j'-1}) = 3$, or $\omega(e_{i+2}) \leq \frac{13}{6}$ when $\omega(e_{j'+1}) = 3$;*
4. *for the edges $e_{j'+1}, e_{j'+2}$ in Fig. 4.14c, $\omega(e_{j'+1}) \leq \frac{13}{6}$ and $\omega(e_{j'+2}) \leq \frac{7}{3}$;*
5. *for the edges $e_{j'-1}, e_{j'-2}, e_{j'+1}, e_{j'+2}$ in Fig. 4.14d, $\omega(e) \leq \frac{35}{12}$.*

Proof. We first note that in items 2) and 5) we do not succeed in getting a better bound, and thus quote the existing bounds. More specifically, for the edge $e_{j'+2}$ in Fig. 4.11c, $\omega(e_{j'+2}) \leq \frac{35}{12}$ is from Lemma 4.14; for the others, $\omega(e) \leq \frac{35}{12}$ is from Lemma 4.15.

In the rest of the proof, we let e_{i_1, j_1}^* denote the edge of $C^*(e_{i,j})$ such that $|C(e_{i_1, j_1}^*)| = 1$.

1. The edge $e_{j'+2}$ in Figs. 4.4, 4.7a and 4.10b.

One sees that for the edge $e_{j'+2}$ in Fig. 4.10b, its $\omega(e_{j'+2})$ is larger (or worst) when $\mathcal{N}_p(e_{j'+2}) = \emptyset$ than when $\mathcal{N}_p(e_{j'+2}) \neq \emptyset$. We therefore consider the worse case when $\mathcal{N}_p(e_{j'+2}) = \emptyset$; this way, all three edges can be discussed exactly the same (ignoring the incidence information of $i - 2$ and $j' - 2$ in M).

Assume the edge $e_{j'+2}$ is incident at h , i.e., $e_{h, j'+2} := e_{j'+2}$. We consider the case where $|C^*(e_{h, j'+2})| \geq 5$, as otherwise $\omega(e_{j'+2}) \leq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} < \frac{29}{12}$. When there is an edge $e_{h_1, \ell_1}^* \in C^*(e_{h, j'+2})$ such that $|C(e_{h_1, \ell_1}^*)| = 1$, then either $h_1 = h + 1$ or $\ell_1 = j' + 3$. If there is an edge e_h^* of M^* incident at h , then $|C(e_h^*)| \geq 3$, since otherwise the algorithm \mathcal{LS} would be able to expand M by swapping the four edges $e_{i,j}, e_{h-1, j'+1}, e_{h, j'+2}, e_{i+2}$ of $C(C^*(e_{i,\cdot}))$ by the five edges $e_h^*, e_{h_1, \ell_1}^*, e_{i_1, j_1}^*, e_{i, j'}, e_{i+1, j'+1}^*$; for the same reason, if there is an edge e_{h-1}^* of M^* incident at $h - 1$, then $|C(e_{h-1}^*)| \geq 3$. These together say that the value combination of $\tau(e_{h, j'+2} \leftarrow C^*(e_{h,\cdot}))$ is impossible to have two values $\geq \frac{1}{2}$. Therefore, if $|C^*(e_{h, j'+2})| = 5$, we have $\omega(e_{h, j'+2}) \leq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} = \frac{29}{12}$, due to Lemmas 4.1, 4.3, and $|C(e_{i+1, j'+1}^*)| = 4$ and $|C(e_{i+2, j'+2}^*)| \geq 3$. When there is no edge $e^* \in C^*(e_{h, j'+2})$ such that $|C(e^*)| = 1$, if $|C^*(e_{h, j'+2})| = 5$, then we have $\omega(e_{h, j'+2}) \leq 4 \times \frac{1}{2} + \frac{1}{4} < \frac{29}{12}$, due to $|C(e_{i+1, j'+1}^*)| = 4$.

We next consider $|C^*(e_{h, j'+2})| = 6$ and $C^*(e_{h, j'+2}) = \{e_{i+1, j'+1}^*, e_{i+2, j'+2}^*, e_{i+3, j'+3}^*, e_{h-1, \ell-1}^*, e_{h, \ell}^*, e_{h+1, \ell+1}^*\}$, for some ℓ . If there is an edge of $C^*(e_{h, j'+2})$ conflicts only one edge of M , then this edge has to be $e_{h+1, \ell+1}^*$. Then we have $|C(e_{i+2, j'+2}^*)| \geq 4$, since

otherwise the algorithm \mathcal{LS} would replace the four edges $e_{i,j}$, e_{i+2} , $e_{h-1,j'+1}$, $e_{h,j'+2}$ by the five edges e_{i_1,j_1}^* , $e_{i,j'}^*$, $e_{i+1,j'+1}^*$, $e_{i+2,j'+2}^*$, $e_{h+1,\ell+1}^*$ to expand M , a contradiction. For a similar reason, we have $|C(e_{h,\ell}^*)| \geq 3$ and then $|C(e_{h-1,\ell-1}^*)| \geq 4$. It follows that $|C(e_{i+3,j'+3}^*)| \geq 3$ and $|C(e_{h,\ell}^*)| = 3$. Therefore, we have $\omega(e_{h,j'+2}) \leq (2 \times \frac{1}{4} + \frac{1}{3}) + (1 + \frac{1}{3} + \frac{1}{4}) = \frac{29}{12}$. When there is no edge of $C^*(e_{h,j'+2})$ conflicts only one edge of M , since we cannot have both $|C(e_{h,\ell}^*)| = |C(e_{h-1,\ell-1}^*)| = 2$, we have $\omega(e_{h,j'+2}) \leq (\frac{1}{4} + \frac{1}{3} + \frac{1}{2}) + (2 \times \frac{1}{2} + \frac{1}{3}) = \frac{29}{12}$.

Note that in the above proof we did not use the incidence information of $i-2$ and $j'-2$ in M . In summary, we have $\omega(e_{j'+2}) \leq \frac{29}{12} \approx 2.417$ in Figs. 4.4, 4.7a and 4.10b.

3. The edges e_{i-2} and e_{i+2} in Fig. 4.14b, which can be discussed exactly the same.

Recall that there is an edge $e_{i_1,j_1}^* \in C^*(e_{.,j})$ such that $|C(e_{i_1,j_1}^*)| = 1$.

From Lemma 4.15, we know that when $\omega(e_{j'-1}) = 3$, e_{i-2} has to be a parallel edge of M ; if e_{i-2} is a singleton then $\omega(e_{i-2}) \leq \frac{35}{12}$, and if $\omega(e_{j'-1}) < 3$, then $\omega(e_{j'-1}) \leq \frac{35}{12}$. We consider in the following $\omega(e_{j'-1}) = 3$.

Assume the edge $e_{j'-1}$ is incident at h' , i.e., $e_{h',j'-1} := e_{j'-1}$. Thus we have $|C(e_{i-2,j'-2}^*)| = 3$ (that is, no edge of M incident at $j'-3$), $|C(C^*(e_{.,j'-1}))| = 5$, and there is an edge $e_{h'_1,j'_1}^* \in C^*(e_{h',.})$ such that $|C(e_{h'_1,j'_1}^*)| = 1$.

Assume the edge e_{i-2} is incident at ℓ , i.e., $e_{i-2,\ell} := e_{i-2}$. We observe first that if there is an edge e_ℓ^* of M^* incident at ℓ , then $|C(e_\ell^*)| \geq 3$, since otherwise the algorithm \mathcal{LS} would be able to expand M by swapping the five edges of $C(C^*(e_{.,j'-1}))$ by six edges including e_ℓ^* ; for the same reason, if there is an edge $e_{\ell-1}^*$ of M^* incident at $\ell-1$, then $|C(e_{\ell-1}^*)| \geq 3$; if there is an edge $e_{i-3}^* = e_{i-3,j'-3}^*$ of M^* incident at $i-3$, then $|C(e_{i-3}^*)| \geq 3$. This says that the value combination of $\tau(e_{i-2,\ell} \leftarrow C^*(e_{i-2,\ell}))$ is impossible to have two values $\geq \frac{1}{2}$.

If there is an edge of $C^*(e_{i-2,\ell})$ conflicting only one edge of M , then this edge has to be $e_{\ell+1}^*$. In this case, the algorithm \mathcal{LS} would replace the five edges of $C(C^*(e_{.,j'-1}))$ by the edges $e_{\ell+1}^*$, $e_{h'_1,j'_1}^*$, e_{i_1,j_1}^* and the three edges of $C^*(e_{.,j'-1})$ to expand M , a contradiction. Therefore, there is no edge of $C^*(e_{i-2,\ell})$ conflicting only one edge of M . It follows that if $|C^*(e_{i-2,\ell})| \leq 5$, we have $\omega(e_{i-2,\ell}) \leq \frac{1}{2} + 4 \times \frac{1}{3} = \frac{11}{6}$.

We next assume $C^*(e_{i-2,\ell}) = \{e_{i-1,j'-1}^*, e_{i-2,j'-2}^*, e_{i-3,j'-3}^*, e_{h-1,\ell-1}^*, e_{h,\ell}^*, e_{h+1,\ell+1}^*\}$, for some h . Note that every one of $e_{i-3,j'-3}^*$, $e_{h-1,\ell-1}^*$, $e_{h,\ell}^*$ conflicts both edges $e_{i-3,\ell-1}$ and $e_{i-2,\ell}$. If there is one of them conflicting only these two edges of M , then the five edges of $C(C^*(e_{.,j'-1}))$ can be replaced by six edges to expand M . It thus follows that all three $|C(e_{i-3,j'-3}^*)|$, $|C(e_{h-1,\ell-1}^*)|$, $|C(e_{h,\ell}^*)| \geq 3$; and subsequently $\omega(e_{i-2,\ell}) \leq 5 \times \frac{1}{3} + \frac{1}{2} = \frac{13}{6} \approx 2.167$.

In summary, we have for e_{i-2} in Fig. 4.14b: if $\omega(e_{j'-1}) \leq \frac{35}{12}$ then $\omega(e_{i-2}) \leq \frac{35}{12}$; if $\omega(e_{j'-1}) = 3$ then $\omega(e_{i-2}) \leq \frac{13}{6}$. Similarly, we have for e_{i+2} in Fig. 4.14b: if $\omega(e_{j'+1}) \leq \frac{35}{12}$ then $\omega(e_{i+2}) \leq \frac{35}{12}$; if $\omega(e_{j'+1}) = 3$ then $\omega(e_{i+2}) \leq \frac{13}{6}$.

- 4.1. The edge $e_{j'+1}$ in Fig. 4.14c.

Recall that there is an edge $e_{i_1,j_1}^* \in C^*(e_{.,j})$ such that $|C(e_{i_1,j_1}^*)| = 1$.

From Lemma 4.15, we know that if $\omega(e_{j'-1}) < 3$, then $\omega(e_{j'-1}) \leq \frac{35}{12}$. We consider in the following $\omega(e_{j'-1}) = 3$. Assume the edge $e_{j'-1}$ is incident at h' , i.e., $e_{h',j'-1} := e_{j'-1}$. From $\omega(e_{j'-1}) = 3$, there is an edge $e_{h'_1,j'_1}^* \in C^*(e_{h',.})$ such that $|C(e_{h'_1,j'_1}^*)| = 1$.

Assume the edge $e_{j'+1}$ is incident at h , i.e., $e_{h,j'+1} := e_{j'+1}$. We observe first that if there is an edge e_h^* of M^* incident at h , then $|C(e_h^*)| \geq 3$, since otherwise the algorithm \mathcal{LS} would be able to expand M by swapping the five edges of $C(C^*(e_{i.,}))$ by six edges including e_h^* ; for the same reason, if there is an edge e_{h+1}^* of M^* incident at $h+1$, then $|C(e_{h+1}^*)| \geq 3$. This says that the value combination of $\tau(e_{h,j'+2} \leftarrow C^*(e_{h.,}))$ is

impossible to have two values $\geq \frac{1}{2}$.

If there is an edge of $C^*(e_{h,j'+1})$ conflicting only one edge of M , then this edge has to be e_{h-1}^* . In this case, the algorithm \mathcal{LS} would replace the five edges of $C(C^*(e_{i,\cdot}))$ by the edges $e_{h-1}^*, e_{h'_1,j'_1}^*, e_{i_1,j_1}^*$ and the three edges of $C^*(e_{i,\cdot})$ to expand M , a contradiction. Therefore, there is no edge of $C^*(e_{h,j'+1})$ conflicting only one edge of M . It follows that if $|C^*(e_{h,j'+1})| \leq 5$, we have $\omega(e_{h,j'+1}) \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} = 2$.

We next assume $C^*(e_{h,j'+1}) = \{e_{h-1,\ell-1}^*, e_{h,\ell}^*, e_{h+1,\ell+1}^*, e_{i,j'}^*, e_{i+1,j'+1}^*, e_{i+2,j'+2}^*\}$, for some ℓ . Note that every one of $e_{i+2,j'+2}^*, e_{h,\ell}^*, e_{h+1,\ell+1}^*$ conflicts both edges $e_{h,j'+1}$ and $e_{h+1,j'+2}$. If there is one of them conflicting only these two edges of M , then the five edges of $C(C^*(e_{i,\cdot}))$ can be replaced by six edges to expand M . It thus follows that all three $|C(e_{i+2,j'+2}^*)|, |C(e_{h,\ell}^*)|, |C(e_{h+1,\ell+1}^*)| \geq 3$; and subsequently $\omega(e_{h,j'+1}) \leq 5 \times \frac{1}{3} + \frac{1}{2} = \frac{13}{6} \approx 2.167$.

From Lemma 4.15, we also know that if $\omega(e_{i-2}) < 3$, then $\omega(e_{i-2}) \leq \frac{35}{12}$. We consider $\omega(e_{i-2}) = 3$, which implies that there is an edge $e_{h'_1,j'_1}^* \in C^*(e_{i-2})$ such that $|C(e_{h'_1,j'_1}^*)| = 1$. It follows by the same argument as in the above that $\omega(e_{j'+1}) \leq \frac{13}{6}$.

In summary, we have for $e_{j'+1}$ in Fig. 4.14c: if both $\omega(e_{j'-1}), \omega(e_{i-2}) \leq \frac{35}{12}$ then $\omega(e_{j'+1}) \leq \frac{35}{12}$ too; otherwise, $\omega(e_{j'+1}) \leq \frac{13}{6}$.

4.2. The edge $e_{j'+2}$ in Fig. 4.14c.

The argument here is the same as in the last item (4.1) to consider $\omega(e_{j'-1}) = 3$.

Assume the edge $e_{j'+2}$ is incident at h , i.e., $e_{h,j'+2} := e_{j'+2}$.

We observe first, for the same reasons, that if there is an edge e_h^* of M^* incident at h , then $|C(e_h^*)| \geq 3$; if there is an edge e_{h+1}^* of M^* incident at $h+1$, then $|C(e_{h+1}^*)| \geq 3$; there is no edge of $C^*(e_{h,j'+2})$ conflicting only one edge of M ; if there is an edge $e_{j'+2}^*$ of M^* incident at $j'+2$, then $|C(e_{j'+2}^*)| \geq 2$; if there is an edge $e_{j'+3}^*$ of M^* incident at $j'+3$, then $|C(e_{j'+3}^*)| \geq 2$. These together say that the value combination of $\tau(e_{h,j'+2} \leftarrow C^*(e_{h,j'+2}))$ is impossible to have a value 1, and it is impossible to have three values $\geq \frac{1}{2}$. It follows that $\omega(e_{h,j'+2}) \leq 2 \times \frac{1}{2} + 4 \times \frac{1}{3} = \frac{7}{3} \approx 2.333$.

In summary, we have for $e_{j'+2}$ in Fig. 4.14c: if both $\omega(e_{j'-1}), \omega(e_{i-2}) \leq \frac{35}{12}$ then $\omega(e_{j'+2}) \leq \frac{35}{12}$ too; otherwise, $\omega(e_{j'+2}) \leq \frac{7}{3}$.

This finishes the proof of the lemma. \blacktriangleleft

Lemma 4.5 states the 12 possible value combinations of $\tau(e_{i,j} \leftarrow C^*(e_{i,\cdot}))$ with $\omega(e_{i,j}) \geq 3$, which are $\{1, \frac{1}{2}, \frac{1}{2}\}, \{1, \frac{1}{2}, \frac{1}{3}\}, \{1, \frac{1}{2}, \frac{1}{4}\}, \{1, \frac{1}{3}, \frac{1}{3}\}, \{\frac{1}{2}, \frac{1}{2}, \frac{1}{3}\}, \{\frac{1}{2}, \frac{1}{2}, \frac{1}{4}\}, \{\frac{1}{2}, \frac{1}{3}, \frac{1}{3}\}, \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}, \{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\}, \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$, and $\{\frac{1}{2}, \frac{1}{2}, 0\}$. We next count the minimum number of known-to-be parallel edges of M in $C(C^*(e_{i,\cdot}))$ and use Lemmas 4.19 and 4.20 to upper bound their $\omega(\cdot)$ values respectively, for each combination.

1. $\{1, \frac{1}{2}, \frac{1}{2}\}$: there are at least 2 parallel edges of M , each with $\omega(\cdot) \leq \frac{5}{2} = 2.5$ (by Lemma 4.19);
2. $\{1, \frac{1}{2}, \frac{1}{3}\}$: there is at least 1 parallel edge of M , with $\omega(\cdot) \leq \frac{5}{2} = 2.5$ (by Lemma 4.19);
3. $\{1, \frac{1}{2}, \frac{1}{4}\}$: there are at least 2 parallel edges of M , each with $\omega(\cdot) \leq \frac{29}{12} \approx 2.417$ (by Lemmas 4.19, 4.20);
4. $\{1, \frac{1}{3}, \frac{1}{3}\}$: no parallel edge;
5. $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{3}\}$: there are at least 2 parallel edges of M , each with $\omega(\cdot) \leq \frac{5}{2} = 2.5$ (by Lemma 4.19);
6. $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{4}\}$: there are at least 2 parallel edges of M , each with $\omega(\cdot) \leq \frac{29}{12} \approx 2.417$ (by Lemmas 4.19, 4.20);
7. $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{3}\}$: there is at least 1 parallel edge of M , with $\omega(\cdot) \leq \frac{7}{3} \approx 2.333$ (by Lemma 4.19);
8. $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$: there are three possible cases,

- a. there is at least 1 parallel edge of M with $\omega(\cdot) \leq \frac{7}{3} \approx 2.333$ (by Lemma 4.19), or
- b. there are at least 2 parallel edges of M , each with $\omega(\cdot) \leq \frac{29}{12} \approx 2.417$ (by Lemmas 4.19, 4.20), or
- c. there are at least 2 parallel edges of M , one with $\omega(\cdot) \leq \frac{9}{4} = 2.25$ and the other with $\omega(\cdot) \leq \frac{35}{12} \approx 2.917$ (by Lemmas 4.19, 4.20);
9. $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\}$: there are at least 2 parallel edges of M , each with $\omega(\cdot) \leq \frac{11}{5} = 2.2$ (by Lemma 4.19);
10. $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$: no parallel edge;
11. $\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$: there are five possible cases,
 - a. there is no singleton edge other than $e_{i,j}$ with $\omega(\cdot) \geq 3$, no parallel edge;
 - b. there is no singleton edge other than $e_{i,j}$ with $\omega(\cdot) \geq 3$, but there is at least 1 parallel edge of M with $\omega(\cdot) \leq \frac{35}{12} \approx 2.917$ (by Lemmas 4.19, 4.20),
 - c. there is no singleton edge other than $e_{i,j}$ with $\omega(\cdot) \geq 3$, but there are at least 2 parallel edges of M , each with $\omega(\cdot) \leq \frac{35}{12} \approx 2.917$ (by Lemmas 4.19, 4.20),
 - d. there is one singleton edge other than $e_{i,j}$ with $\omega(\cdot) = 3$, accompanied by at least 1 parallel edge of M with $\omega(\cdot) \leq \frac{13}{6} \approx 2.167$ (by Lemma 4.20),
 - e. there is one singleton edge other than $e_{i,j}$ with $\omega(\cdot) = 3$, accompanied by at least 2 parallel edges of M , one with $\omega(\cdot) \leq \frac{13}{6} \approx 2.167$ and the other with $\omega(\cdot) \leq \frac{7}{3} \approx 2.333$ (by Lemma 4.20);
12. $\{\frac{1}{2}, \frac{1}{2}, 0\}$: no parallel edge.

Lemma 4.4 states the 8 possible value combinations of $\tau(e_{i,j} \leftarrow C^*(e_{i,j}))$ with $\omega(e_{i,j}) \geq 3$, which are $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}\}$, $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}\}$, $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}\}$, $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$, $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}\}$, $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$, $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$, and $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\}$. These combinations give rise to $\omega(e_{i,j}) = \frac{10}{3}, \frac{13}{4}, \frac{19}{6}, \frac{37}{12}, \frac{91}{30}, 3, 3$ and 3 respectively. Based on the above list, we conclude the minimum number of known-to-be parallel edges of M in $C(C^*(e_{i,j}))$ for each combination, using the cut-off upper bound 2.5 on their $\omega(\cdot)$ values, as follows.

1. $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}\}$ ($\omega(e_{i,j}) = \frac{10}{3}$): there are at least 4 parallel edges of M ;
2. $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}\}$ ($\omega(e_{i,j}) = \frac{13}{4}$): there are at least 4 parallel edges of M ;
3. $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}\}$ ($\omega(e_{i,j}) = \frac{19}{6}$): there are at least 3 parallel edges of M ;
4. $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$ ($\omega(e_{i,j}) = \frac{37}{12}$): there are at least 3 parallel edges of M ;
5. $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}\}$ ($\omega(e_{i,j}) = \frac{91}{30}$): there are at least 4 parallel edges of M ;
6. $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$ ($\omega(e_{i,j}) = 3$): there are at least 2 parallel edges of M ;
7. $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ ($\omega(e_{i,j}) = 3$): there are two possible cases,
 - a. there is no singleton edge other than $e_{i,j}$ with $\omega(\cdot) \geq 3$, but there are at least 2 parallel edges of M ;
 - b. there is one singleton edge other than $e_{i,j}$ with $\omega(\cdot) = 3$, accompanied by at least 3 parallel edges of M ;
8. $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\}$ ($\omega(e_{i,j}) = 3$): there are at least 2 parallel edges of M .

We conclude this section with the following lemma.

- **Lemma 4.21.** *Every edge $e_{i,j} \in M$ with $\omega(e_{i,j}) \geq 3$ must be a singleton, and $\omega(e_{i,j}) \in \{\frac{10}{3}, \frac{13}{4}, \frac{19}{6}, \frac{37}{12}, \frac{91}{30}, 3\}$. Furthermore,*
1. *the existence of an edge $e_{i,j} \in M$ with $\omega(e_{i,j}) = \frac{10}{3}$ or $\frac{13}{4}$ or $\frac{91}{30}$ is accompanied with at least 4 parallel edges of M each with $\omega(\cdot) \leq 2.5$;*
 2. *the existence of an edge $e_{i,j} \in M$ with $\omega(e_{i,j}) = \frac{19}{6}$ or $\frac{37}{12}$ is accompanied with at least 3 parallel edges of M each with $\omega(\cdot) \leq 2.5$;*

3. the existence of an edge $e_{i,j} \in M$ with $\omega(e_{i,j}) = 3$ is accompanied with at least 1.5 parallel edges of M each with $\omega(\cdot) \leq 2.5$.

Each of these accompanying parallel edges must belong to either $C(C^*(e_{i,\cdot}))$ with $|C^*(e_{i,\cdot})| = 3$, or $C(C^*(e_{\cdot,j}))$ with $|C^*(e_{\cdot,j})| = 3$, for some $e_{i,j}$.

4.7 An upper bound on the average value of $\omega(e)$

Let $M_{\geq 3}$ be the subset of all the edges of M with $\omega(\cdot) \geq 3$, and let $n_s = |M_{\geq 3}|$. Let P denote the subset of all the accompanying parallel edges of M determined in Lemma 4.21, and let $n_p = |P|$. From Lemma 4.10, every edge of $M_{\geq 3}$ is a singleton, and thus $M_{\geq 3} \cap P = \emptyset$.

► **Lemma 4.22.** *Each edge of P belongs to $C(C^*(e_{i,j}))$ for at most four distinct edges $e_{i,j} \in M_{\geq 3}$.*

Proof. Consider an edge $e_{h,\ell} \in P$ and assume that the edge $e_{h+1,\ell+1}$ is also in M .

Consider the vertex d_ℓ^B at which $e_{h,\ell}$ is incident; let $e_{\ell-2}^*, e_{\ell-1}^*, e_\ell^*, e_{\ell+1}^*, e_{\ell+2}^*$ be the edge of M^* incident at the vertex $d_{\ell-2}^B, d_{\ell-1}^B, d_\ell^B, d_{\ell+1}^B, d_{\ell+2}^B$, respectively, if such an edge exists. Clearly, for any edge $e_{i,j} \in M_{\geq 3}$, if $C^*(e_{i,j})$ does not contain any of the five edges $e_{\ell-2}^*, e_{\ell-1}^*, e_\ell^*, e_{\ell+1}^*, e_{\ell+2}^*$, then $e_{h,\ell} \notin C(C^*(e_{i,j}))$ (unless $e_{h,\ell} \in C(C^*(e_{i,j}))$ through the symmetric discussion using the vertex d_h^A). We distinguish two cases where $C^*(e_{i,\cdot})$ contains one of the five edges and $C^*(e_{\cdot,j})$ contains one of the five edges, respectively.

When $C^*(e_{i,\cdot})$ contains one of the five edges $e_{\ell-2}^*, e_{\ell-1}^*, e_\ell^*, e_{\ell+1}^*, e_{\ell+2}^*$, we see from all the 27 configurations of $C(C^*(e_{i,\cdot}))$ and Lemma 4.15 that neither of the edges $e_\ell^*, e_{\ell+1}^*$, if exists, can be incident at the vertex d_i^A . It follows that the vertex d_i^A is an end of one of the three edges $e_{\ell-2}^*, e_{\ell-1}^*, e_{\ell+2}^*$.

When $C^*(e_{\cdot,j})$ contains one of the five edges $e_{\ell-2}^*, e_{\ell-1}^*, e_\ell^*, e_{\ell+1}^*, e_{\ell+2}^*$, we know that $j = \ell - 2$ due to the fact that the edge $e_{i,j}$ is a singleton edge of M .

Since no two edges of $M_{\geq 3}$ are adjacent to a common edge of M^* , we conclude that there are at most three distinct edges $e_{i,j} \in M_{\geq 3}$ such that $e_{h,\ell} \in C(C^*(e_{i,j}))$ through the vertex d_ℓ^B . Furthermore, if there are such three distinct edges, then one is incident at $v_{\ell-2}^B$, one is adjacent to $e_{\ell-1}^*$ (but not incident at $v_{\ell-1}^B$), and the other is adjacent to $e_{\ell+2}^*$ (but not incident at $v_{\ell+2}^B$); the five edges $e_{\ell-2}^*, e_{\ell-1}^*, e_\ell^*, e_{\ell+1}^*, e_{\ell+2}^*$ all exist, so do the extra two edges $e_{\ell-3}^*$ and $e_{\ell+3}^*$, and these seven edges of M^* are consecutively parallel.

The three edges $e_{\ell-1}^*, e_\ell^*, e_{\ell+1}^*$ of M^* are conflicting with only the three edges of M_{\geq} and the two parallel edges $e_{h,\ell}, e_{h+1,\ell+1}$ of M ; and for each of these three edges of M_{\geq} , there is another distinct edge of M^* conflicting with only this edge of M_{\geq} . In other words, there are six edges of M^* conflicting with only the three edges of M_{\geq} and the two parallel edges $e_{h,\ell}, e_{h+1,\ell+1}$ of M , a contradiction as the algorithm \mathcal{LS} would swap them to expand M .

This proves that there are at most two distinct edges $e_{i,j} \in M_{\geq 3}$ such that $e_{h,\ell} \in C(C^*(e_{i,j}))$ through the vertex d_ℓ^B . Symmetrically, we can prove that there are at most two distinct edges $e_{i,j} \in M_{\geq 3}$ such that $e_{h,\ell} \in C(C^*(e_{i,j}))$ through the vertex d_h^A . Therefore, there are at most four distinct edges $e_{i,j} \in M_{\geq 3}$ such that $e_{h,\ell} \in C(C^*(e_{i,j}))$. ◀

Using Lemma 4.21, assume there is a fraction of xn_s edges of $M_{\geq 3}$ each accompanied with 4 parallel edges of P ; there is a fraction of yn_s edges of $M_{\geq 3}$ each accompanied with 3 parallel edges of P ; and there is a fraction of $(1 - x - y)n_s$ edges of $M_{\geq 3}$ each accompanied with 1.5 parallel edges of P , where $x \geq 0, y \geq 0, 1 - x - y \geq 0$. From Lemma 4.22, we have

$$4n_p \geq 4xn_s + 3yn_s + 1.5(1 - x - y)n_s = (1.5 + 2.5x + 1.5y)n_s,$$

which gives

$$\frac{n_p}{n_s} \geq \frac{1.5 + 2.5x + 1.5y}{4}, \quad (9)$$

and the average amount of tokens for all the edges of $M_{\geq 3} \cup P$ is, using Equation 9,

$$\overline{\omega(e)} \leq \frac{2.5n_p + \frac{10}{3}xn_s + \frac{19}{6}yn_s + 3(1-x-y)n_s}{n_p + n_s} \leq \frac{5}{2} + \frac{12+8x}{33+15x} \leq \frac{35}{12}. \quad (10)$$

Lemma 4.15 tells that every other edge of M has its $\omega(\cdot) \leq \frac{35}{12}$ too. Therefore, the average amount of tokens for all the edges of M is no greater than $\frac{35}{12}$. We have thus proved the following theorem.

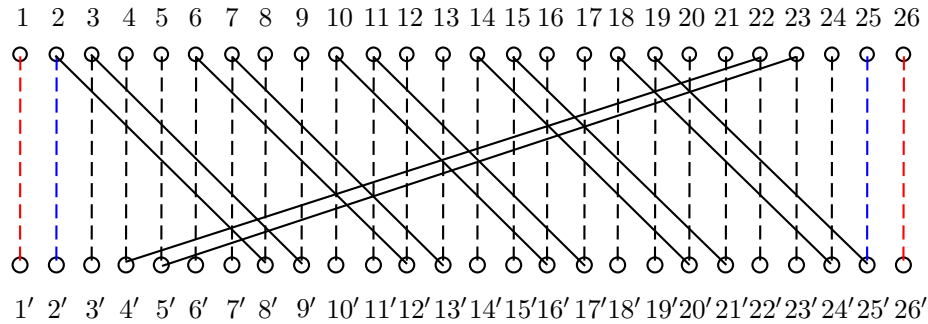
► **Theorem 4.23.** *The algorithm \mathcal{LS} is an $O(n^{13})$ -time $\frac{35}{12}$ -approximation for both the MCBM and the MAX-DUO problems.*

5 Lower bounds on the locality gap for the algorithm \mathcal{LS}

In this section, we present two instances of the MCBM and MAX-DUO problems, respectively, to show that the approximation ratio of the algorithm \mathcal{LS} has a lower bound of $\frac{13}{6} > 2.166$ for MCBM and a lower bound of $\frac{5}{3} > 1.666$ for MAX-DUO.

5.1 An instance of MCBM

Consider the bipartite graph $G = (V^A, V^B, E)$ shown in Fig. 5.1, where $V^A = \{1, 2, \dots, 26\}$, $V^B = \{1', 2', \dots, 26'\}$, and E is the set of all the edges in solid and dashed lines. One can see that the set of 26 consecutive parallel edges (in dashed lines) is an optimal solution M^* to the MCBM problem on G . Let M be the maximal compatible matching shown as solid lines in Fig. 5.1, and assume it is the starting matching for the algorithm \mathcal{LS} on G .



■ **Figure 5.1** A bipartite graph $G = (V^A, V^B, E)$, where $V^A = \{1, 2, \dots, 26\}$, $V^B = \{1', 2', \dots, 26'\}$, and $E = M \cup M^*$. M consists of the 12 edges in solid lines; it is a maximal compatible matching in G ; M^* consists of the 26 edges in dashed lines; it is an optimal compatible matching to the MCBM problem on G .

► **Lemma 5.1.** *M cannot be further improved by the algorithm \mathcal{LS} due to the following reasons:*

1. M is a maximal compatible matching in G ;
2. all the edges of M are parallel edges;
3. for any $e \in M$, there is at most one edge of M^* compatible with all the edges in $M - \{e\}$;

4. for any 2 edges $e_1, e_2 \in M$, there are at most two edges of M^* compatible with all the edges in $M - \{e_1, e_2\}$;
5. for any 3 edges $e_1, e_2, e_3 \in M$, there are at most three edges in M^* compatible with all the edges in $M - \{e_1, e_2, e_3\}$;
6. for any 4 edges $e_1, \dots, e_4 \in M$, there are at most four edges in M^* compatible with all the edges in $M - \{e_1, \dots, e_4\}$;
7. for any 5 edges $e_1, \dots, e_5 \in M$, there are at most five edges in M^* compatible with all the edges in $M - \{e_1, \dots, e_5\}$.

Proof. The first two items are trivial. We note that the second item implies that M cannot be further improved by the algorithm using the operation REDUCE-5-BY-5. We next show that M cannot be further improved by the algorithm using the operation REPLACE-5-BY-6.

We examine whether some edges of M can be swapped out for more edges from M^* by REPLACE-5-BY-6. For ease of presentation, we partition M^* into 3 subsets $M^{*1} = \{(1, 1'), (26, 26')\}$, $M^{*2} = \{(2, 2'), (25, 25')\}$, and $M^{*3} = \{(3, 3'), (4, 4'), \dots, (24, 24')\}$ (in Fig. 5.1, their edges are colored red, blue, black, respectively). We have the following observations.

► **Observation 5.1.** To swap for an edge of M^{*1} , a unique edge of M has to be swap out. In details, $(2, 8')$ needs to be swapped out for $(1, 1')$, and $(19, 25')$ needs to be swapped out for $(26, 26')$.

► **Observation 5.2.** To swap for an edge of M^{*2} , a unique pair of parallel edges of M has to be swap out. In details, $(2, 8')$ and $(3, 9')$ need to be swapped out for $(2, 2')$, and $(19, 25')$ and $(18, 24')$ need to be swapped out for $(25, 25')$.

► **Observation 5.3.** To swap for an edge of M^{*3} , a unique triplet of edges of M has to be swap out. In details, when $i = 3 \pmod{4}$, e_{i-1}, e_i, e_{i+1}' need to be swapped out for (i, i') ; when $i = 0 \pmod{4}$, $e_{i-1}, e_{i'}, e_{i+1}'$ need to be swapped out for (i, i') ; when $i = 1 \pmod{4}$, $e_{i-1}', e_{i'}, e_{i+1}$ need to be swapped out for (i, i') ; when $i = 2 \pmod{4}$, e_{i-1}', e_i, e_{i+1} need to be swapped out for (i, i') .

The Observations 5.1, 5.2, 5.3 prove trivially the items 3–5 of the lemma.

To prove the item 6, we next see what a subset of four edges of M can do. If these four edges are able to swap in one edge of M^{*3} , then by Observation 5.3 this edge of M^{*3} actually requires three out of the four edges. Note that these particular three edges are not able to swap for any other edge of M^{*3} . If they contain a unique pair of parallel edges of M in Observation 5.2, then they can swap in three edges one from each of M^{*1}, M^{*2}, M^{*3} , and the fourth edge either forms with two of them to form another triplet to swap in an other edge of M^{*3} , or it is able to swap in the other edge of M^{*1} . If these particular three edges do not contain a unique pair of parallel edges of M in Observation 5.2, then they can swap in only the edge of M^{*3} , and the fourth edge can either form with one of them to form a unique pair of parallel edges of M in Observation 5.2 to swap in two edges one from each of M^{*1}, M^{*2} , and/or form with two of them to form another triplet to swap in an other edge of M^{*3} , or it is able to swap in the other edge of M^{*1} . Therefore, these four edges can swap in the best case four edges $(1, 1'), (2, 2'), (3, 3')$ (or $(26, 26'), (25, 25'), (24, 24')$, respectively) and another edge of $M^{*1} \cup M^{*3}$. If these four edges are not able to swap in any edge of M^{*3} , then they can swap in the best case four edges of $M^{*1} \cup M^{*2}$.

To prove the item 7, we next see what a subset of five edges of M can do. If these five edges are able to swap in two edges of M^{*3} , then by Observation 5.3 these two edges of M^{*3} actually require at least four out of the five edges. Note that these particular four edges are

only able to swap for four edges of M^* , including the above two edges of M^{*3} and the other two edges must be either $(1, 1')$, $(2, 2')$ or $(26, 26')$, $(25, 25')$. Then the fifth edge either forms with two of these particular four edges to form another triplet to swap in an other edge of M^{*3} , or it is able to swap in the other edge of M^{*1} . If these five edges are not able to swap in at least two edges of M^{*3} , then they can swap in the best case one edge of M^{*3} and four edges of $M^{*1} \cup M^{*2}$. ◀

► **Theorem 5.2.** *There is a lower bound of $\frac{13}{6} > 2.166$ on the locality gap of the algorithm \mathcal{LS} for the MCBM problem.*

Proof. By Lemma 5.1, if the matching M is fed as the starting matching to the algorithm \mathcal{LS} , then the algorithm terminates without modifying it. Note that we have $|M| = 12$ and $|M^*| = 26$, we conclude that the algorithm can not do better than $\frac{13}{6}$ in the worst case. ◀

► **Remark.** Using our amortized analysis, let $C(e^*)$ be the subset of edges of M conflicting with the edge $e^* \in M^*$. Then in the above instance, we have $|C(e^*)| = 1, 2, 3$ for $e^* \in M^{*1}, M^{*2}, M^{*3}$, respectively. The maximum total amount of tokens received by the edges of M is achieved at $(2, 8')$, where $\omega((2, 8')) = 1 + \frac{1}{2} + 4 \times \frac{1}{3} = \frac{17}{6} \approx 2.833$. This maximum is quite close to the approximation ratio $\frac{35}{12} \approx 2.917$ of the algorithm \mathcal{LS} , which is also the maximum possible $\omega(\cdot)$ value for the parallel edges of M . (Recall that $\frac{35}{12} = 1 + 2 \times \frac{1}{2} + 2 \times \frac{1}{3} + \frac{1}{4}$.)

► **Remark.** We may have variants of the algorithm \mathcal{LS} by substituting the operation REPLACE-5-BY-6 with the similarly defined operation REPLACE- ρ -BY- $(\rho + 1)$, for any $\rho \geq 1$ (with or without the operation REDUCE-5-BY-6).

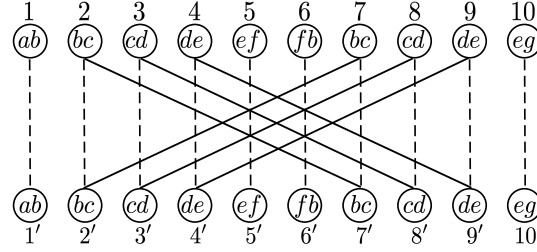
For $\rho = 1, 2, 3, 4$, we can construct similar instances to show the corresponding lower bounds on the locality gap for the variants on the MCBM problem.

- $\rho = 4$.
We can construct a similar instance $G = (V^A, V^B, E)$, except that $|V^A| = |V^B| = 22$ and $|M| = 10$. The performance ratio of the algorithm is $\frac{11}{5} = 2.2$.
- $\rho = 3$.
We can construct two different instances for $G = (V^A, V^B, E)$. One is similar to the previous two, except that $|V^A| = |V^B| = 18$ and $|M| = 8$. In the other we have $|V^A| = |V^B| = 9$ and $|M| = 4$, such that the four edges of M are consecutively parallel. The performance ratio of the algorithm on both instances is $\frac{9}{4} = 2.25$.
- $\rho = 2$.
We can construct an instance similar to the second instance when $\rho = 3$, which is a graph $G = (V^A, V^B, E)$ with $|V^A| = |V^B| = 8$ and $|M| = 3$, such that the three edges of M are consecutively parallel. The performance ratio of the algorithm is $\frac{8}{3} \approx 2.667$.
- $\rho = 1$.
We can construct a similar graph $G = (V^A, V^B, E)$ with $|V^A| = |V^B| = 7$ and $|M| = 2$, such that the two edges of M are parallel. The performance ratio of the algorithm is $\frac{7}{2} = 3.5$. This is essentially the 3.5-approximation by Boria *et al.* [2], and the instance shows that the performance ratio is tight.

5.2 An instance of MAX-DUO

In the instance of MAX-DUO, we have two identical length-11 strings $A = (a, b, c, d, e, f, b, c, d, e, g) = B$. We construct the corresponding bipartite graph $G = (V^A, V^B, E)$ (shown in Fig. 5.2), where $V^A = \{1, 2, \dots, 10\}$ and $V^B = \{1', 2', \dots, 10'\}$. Since $A = B$, each pair of duos represented by the vertices i and i' are the same, for $i = 1, 2, \dots, 10$. Thus it is easy to see that there is an optimal solution M^* to MCBM on G , which consists of all the 10 edges

in dashed lines shown in Fig. 5.2. Let M be the compatible matching consisting of the six edges in solid lines in Fig. 5.2.



■ **Figure 5.2** The corresponding bipartite graph $G = (V^A, V^B, E)$ constructed from two identical strings $A = B = (a, b, c, d, e, f, b, c, d, e, g)$, where $V^A = \{1, 2, \dots, 10\}$, $V^B = \{1', 2', \dots, 10'\}$, and $E = M \cup M^*$. M consists of the six edges in solid lines and it is a compatible matching; M^* consists of the ten edges in dashed lines and it is an optimal compatible matching to the MCBM problem on G .

► **Lemma 5.3.** M cannot be further improved by the algorithm \mathcal{LS} due to the following reasons:

1. M is a local maximal compatible matching;
2. all edges of M are parallel edges;
3. for any $e \in M$, there is no edge in M^* compatible with all the 5 edges in $M - \{e\}$;
4. for any 2 edges $e_1, e_2 \in M$, there are at most 2 edges in M^* compatible with all the 4 edges in $M - \{e_1, e_2\}$;
5. for any 3 edges $e_1, e_2, e_3 \in M$, there are at most 2 edges in M^* compatible with all the 3 edges in $M - \{e_1, e_2, e_3\}$;
6. for any 4 edges $e_1, \dots, e_4 \in M$, there are at most 4 edges in M^* compatible with both of the 2 edges in $M - \{e_1, \dots, e_4\}$;
7. for any 5 edges $e_1, \dots, e_5 \in M$, there are 4 edges in M^* compatible with the only edge in $M - \{e_1, \dots, e_5\}$.

Proof. The first three items are trivial. We note that the second item implies that M cannot be further improved by the algorithm using the operation REDUCE-5-BY-5. We partition M into three subsets $M^2 = \{(2, 7'), (7, 2')\}$, $M^3 = \{(3, 8'), (8, 3')\}$, and $M^4 = \{(4, 9'), (9, 4')\}$.

For any $e \in M$, we see that $|C^*(e)| = 6$, implying there are only 4 edges of M^* compatible with e ; this proves the 7th observation.

For the two edges e_1, e_2 in the same part of the partition, we see that $C^*(e_1) = C^*(e_2)$, implying that if one of them is in M , then none of the six edges of $C^*(e_1)$ can be compatible with it. (This proves again the item 3.) Also, we see that the edges $(1, 1'), (6, 6')$ are not compatible with the edges and only these edges in M^2 ; and the edges $(10, 10'), (5, 5')$ are not compatible with the edges and only these edges in M^4 .

To prove the item 4, we see that when the two edges $e_1, e_2 \in M$ are not in the same part, then from the last paragraph there is no edge in M^* compatible with all the 4 edges in $M - \{e_1, e_2\}$; when the two edges $e_1, e_2 \in M$ are in the same part, then either there are two edges in M^* compatible with all the 4 edges in $M - \{e_1, e_2\}$, if this part is M^2 or M^4 , or otherwise there is no edge in M^* compatible with all the 4 edges in $M - \{e_1, e_2\}$.

For the item 5, any three edges $e_1, e_2, e_3 \in M$ cannot take up two separate parts, and therefore there are at most 2 edges in M^* compatible with all the 3 edges in $M - \{e_1, e_2, e_3\}$.

For any 4 edges $e_1, \dots, e_4 \in M$, if they take up two parts, then there are exactly 4 edges of M^* compatible with the 2 edges in $M - \{e_1, \dots, e_4\}$, which belong to the same part; if

they do not take up two parts, then there are at most 2 edges of M^* compatible with the 2 edges in $M - \{e_1, \dots, e_4\}$. This proves the item 6, and completes the proof of the lemma. ◀

► **Theorem 5.4.** *There is a lower bound of $\frac{5}{3} > 1.666$ on the locality gap of the algorithm \mathcal{LS} for the MAX-DUO problem.*

Proof. By Lemma 5.3, if the matching M is fed as the starting matching to the algorithm \mathcal{LS} , then the algorithm terminates without modifying it. Note that we have $|M| = 6$ and $|M^*| = 10$, we conclude that the algorithm can not do better than $\frac{5}{3}$ in the worst case. ◀

6 Conclusions

We studied the MAX-DUO problem, the complement of the well studied MCSP problem. Motivated by an earlier local search algorithm, we presented an improved heuristics \mathcal{LS} for a more general MCBM problem, that uses one operation to increase the cardinality of the solution and another novel operation to reduce the singleton edges in the solution. The heuristics is iterative and has a time complexity $O(n^{13})$, where n is the length of the input strings. Through an amortized analysis, we are able to show that the proposed algorithm \mathcal{LS} has an approximation ratio of at most $35/12 < 2.917$. This improves the current best 3.25-approximation for both problems, and breaks the barrier of 3. In a companion paper, we are able to design a $(1.4 + \epsilon)$ -approximation for 2-MAX-DUO, a restricted version in which every letter of the alphabet occurs at most twice in each input string. Together, we improved all current best approximability results for the MAX-DUO problem.

We also showed that there is a lower bound of $13/6 > 2.166$ and $5/3 > 1.666$ on the locality gap of the algorithm \mathcal{LS} for the MCBM and the MAX-DUO problems, respectively.

We remark that the time complexity of the algorithm \mathcal{LS} can possibly be reduced using appropriate data structures. For the performance ratio, one would likely do a better analysis by examining more cases with large $\omega(\cdot)$ values, which we are looking into. On the other hand, it is interesting to investigate whether or not swapping more edges can lead to a better approximation.

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